integral of complex-valued function against complex variable

Tip to remember: in the first integral, replace \( z \) by \( \tau(t) \) and use the change-of-variable rule \( \text{d}z = \tau'(t)\text{d}t \).

\[
\int_{a}^{b} g(t)\text{d}t \text{ is interpreted as } \int_{a}^{b} g_{1}(t)\text{d}t + i\int_{a}^{b} g_{2}(t)\text{d}t
\]

\( g(t) = g_{1}(t) + ig_{2}(t) \)

\[
\int_{a}^{b} e^{it}\text{d}t = \int_{a}^{b} \cos(t) + i\sin(t)\text{d}t = \sin(t)\bigg|_{a}^{b} - i\cos(t)\bigg|_{a}^{b}
\]

\[= 0 - i(-1 - 1) = 2i\]

In most cases, one is able to find an antiderivative of \( g \), say \( G \).

\[G' = g\]

which means \( G_{1}' = g_{1} \) and \( G_{2}' = g_{2} \).

\[
\int_{a}^{b} g(t)\text{d}t = \int_{a}^{b} (G_{1}' + iG_{2}')\text{d}t = \int_{a}^{b} G_{1}'(t)\text{d}t + i\int_{a}^{b} G_{2}'(t)\text{d}t
\]

\[= \left[ G_{1}(t) + iG_{2}(t) \right]_{a}^{b}
\]

\[= G(t) \bigg|_{a}^{b}.\]

Thus, \( \int_{a}^{b} g(t)\text{d}t = G(t) \bigg|_{a}^{b} \)

which is of the same form as the Fundamental Theorem of Calc.
In the previous example, $e^{it}$ has an antiderivative $\frac{1}{t} e^{it}$.

Why? \[ \frac{d}{dt} \frac{1}{i} e^{it} = e^{it} \]

When $a = t \in \mathbb{R}$, \[ \frac{d}{dt} \frac{1}{i} e^{it} = e^{it} \]

Then \[ \int_{0}^{\pi} e^{it} dt = \frac{1}{i} \left[ e^{it} \right]_{0}^{\pi} = \frac{1}{i} (e^{i\pi} - e^{i0}) = \frac{1}{i} (-1 - 1) = 2i \]

\[ f(z) = \frac{1}{z^2} = \overline{z}^{-2} \]

\[ a = e^{\pi i}, \quad b = -1 \]

$\gamma_1, \gamma_2, \gamma_3$ as indicated on the picture.

To find $\int_{\gamma_1} f(z) dz$, $\int_{\gamma_2} f(z) dz$, $\int_{\gamma_3} f(z) dz$, we first parametrize $\gamma_1, \gamma_2, \gamma_3$:

* $\gamma_1(t) = a + t(b - a), \quad 0 \leq t \leq 1$
  \[ \Rightarrow \gamma_1'(t) = b - a \]

* $\gamma_2(t) = e^{it}, \quad 0 \leq t \leq \pi$
  \[ \Rightarrow \gamma_2'(t) = i e^{it} \]

* $\gamma_3(t) = e^{it}, \quad -\pi \leq t \leq -\pi/4$ \[ \Rightarrow \gamma_3(t) = \frac{e^{i(\pi + \pi/4 - t)}}{e^{i(-3\pi/4 - t)}} = \frac{e^{i(-\pi/4 + t)}}{e^{i(-3\pi/4 - t)}} \]
  \[ \Rightarrow \quad \gamma_3'(t) = -i e^{-(\pi/4 + t)} \]

\[ \int_{\gamma_1} = \int_{0}^{1} \frac{1}{(a + t(b - a))^2} (b - a) dt \quad (\ast) \]
The integrand has antiderivative

\[ \frac{b-a}{b-a} \cdot \frac{-1}{\alpha + t(b-a)} \]

Thus,

\[ I_1 = \left. \frac{b-a}{\alpha + t(b-a)} \right|_{0}^{1} = \frac{b-a}{b-a} \left( \frac{-1}{b-a} + 1 \right) \]

\[ = \frac{b-a}{b-a} \left( \frac{\sqrt{2}}{2} + 1 - i \frac{\sqrt{2}}{2} \right) \]

Warning: to compute (\( * \)), it is tempting to use substitution

\[ u = t(b-a), \quad du = (b-a)dt \]

Then

\[ (\ast) = \int_{0}^{\frac{b-a}{b-a}} \frac{u}{(\alpha + u)^2} \frac{d}{b-a} = \int_{0}^{1} \frac{du}{(\alpha + u)^2} \]

However, one can see that \( b-a \not\in \mathbb{R} \). The integral \((\ast\ast)\) is not an integral over real variables, since \( u \not\in \mathbb{R} \). The path from 0 to \( b-a \) must be understood as the image of the path \( 0 \longrightarrow 1 \) of \( t \) under the mapping \( t \mapsto t(b-a) \)

scale by \( |b-a| \) and

rotate by \( \text{Arg}(b-a) \)

In short, one has to be careful when attempting to use substitution. The (real) interval of integration will change into a complex path. In most cases, this complicates the problem, not simplifying it.

\[ I_2 = \int_{\pi/4}^{\pi/4} \frac{1}{(e^{-ct})^2} \cdot i e^{it} dt = \int_{\pi/4}^{\pi/4} i e^{3it} dt = i \left[ \frac{e^{3it}}{3i} \right]_{\pi/4}^{\pi/4} \]

\[ = \frac{1}{4i} \left( e^{i3\pi/4} - e^{i3\pi/4} \right) \]

\[ = \cdots \]
\[
I_3 = \int_{-\pi}^{\pi/4} \frac{1}{e^{i \theta} - e^{i \theta/2}} \, dt = \int_{-\pi}^{\pi/4} e^{-3(3\pi/4)} \, dt = -\frac{1}{2} i e^{-i (\frac{3\pi}{4} + b)} \left|_{-\pi}^{\pi/4} \right. 
\]

Now that we have defined complex integrals (in the same fashion as real integrals), let's try to interpret its meaning geometrically.

A map \( f : G \subseteq \mathbb{C} \rightarrow \mathbb{C} \) can be viewed as:

- a function: takes a number to a number,
- a 2D transformation: takes a point to a point,
- a vector field: takes a point to a vector (in other words, assigns a vector to a point).

\[
f(x) = u(x) + i v(x) = (u(x), v(x))
\]

Note that

\[
\int f(z) \, dz \neq \int f(z) \cdot \overline{dz}
\]

\[
\text{complex integral} \quad \text{line integral}
\]

because \( f(z) \, dz \) symbolizes \( f(z) \cdot (\overline{z} + iz) \), whereas \( \int f(z) \cdot \overline{dz} \) symbolizes \( (u(z), v(z)) \cdot dz \).

\[
\text{dot product}
\]

Let's look closer to each integral:

\[
\overline{dz} = dx + idy, \quad dz^2 = (dx, dy)
\]
\[ \int_{\gamma} f(x, y) \cdot ds = \int_{\gamma} (u(x, y)dx + v(x, y)dy) = \int_{\gamma} udx + vdy \]

\[ = W[f, \gamma] \]

This is the work done by force field \( f \) along path \( \gamma \).

\[ \int_{\gamma} f(x, y) \cdot ds = \int_{\gamma} (u + iv)(dx + idy) = \int_{\gamma} (udy - vdx) + i \int_{\gamma} (vdx - udy) \]

\[ = W[f, \gamma] + i F[\overline{f}, \gamma] \]

\( W[f, \gamma] \) is the work done by the conjugate field \( \overline{f} = (u, -v) \) along \( \gamma \),

\( F[\overline{f}, \gamma] \) is the flux of \( \overline{f} \) across \( \gamma \).