Reflection: Calculus of single real variable consists of differential calc. and integral calc. Both are founded on:

- algebraic structure of real numbers,
- the notion of limits

Derivative: \( f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h} \)

Integral: \( \int_{a}^{b} f(x) \, dx = \lim_{n \to \infty} \frac{b-a}{n} \sum_{k=1}^{n} f(x_k) \)

To establish calculus for complex variable, one needs algebraic properties (addition, subtraction, multiplication, division) of complex numbers and the notion of limits.

* Algebraic properties of complex numbers:

  The alg. properties (of real numbers) that are necessary to define derivative and integral are:

  - Addition: if \( a, b \in \mathbb{C} \) then \( a+b \in \mathbb{C} \)

    - Neutral element: \( a+0 = 0+a = a \)

    - Additive inverse: for each \( a \in \mathbb{C} \) there is a unique number (called \( -a \)) such that \( a+(-a) = (a)+(-a) = 0 \)

    - Commutativity: \( a+b = b+a \) (order doesn't matter)

    - Associativity: \( a+(b+c) = (a+b)+c \) (grouping differently gives the same result)

  - Multiplication: if \( a, b \in \mathbb{C} \) then \( ab \in \mathbb{C} \)

    - Identity element: \( 1 \cdot a = a \cdot 1 = a \)

    - Unit element: for each \( a \in \mathbb{C} \) there is a unique number (called \( a^{-1} \)) such that \( a^{-1} \cdot a = a \cdot a^{-1} = 1 \)

    - Commutativity: \( ab = ba \)

    - Associativity: \( a(bc) = (ab)c \)

    - Distributivity: \( (a+b)c = ac + bc \)
In other words, real numbers form a **field**.

\( \mathbb{R} \) has another structure: order, which it inherits from \( \mathbb{N} \) (natural numbers) as a consequence \( x^2 \geq 0 \) for all \( x \).

The equation \( x^2 + 1 = 0 \) has no roots in \( \mathbb{R} \).

Note: this phenomenon doesn’t come from the field properties of \( \mathbb{R} \). For example, \( \begin{bmatrix} a & -b \\ b & a \end{bmatrix} : a, b \in \mathbb{R} \) satisfies all field properties.

Neutral element: \( \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \) "0"

Unit element: \( \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \). Note that \( \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \) "1"

About impossibilities:

- Division by 0 \( \Rightarrow \) forbidden by field properties.

  If \( 1/0 = a \in \mathbb{R} \) then for any \( x \in \mathbb{R} \),

  \[ x = x \cdot 1 = x (0a) = (x0) a = 0a = 1 \]

  \( \Rightarrow \) every number would be equal to 1 (contradiction)

- How impossibilities are fixed by expanding the set of numbers:

  \( \mathbb{N} \xrightarrow{\text{adopt } -1} \mathbb{Z} \xrightarrow{\text{adopt } \sqrt{2} , \sqrt{3} , \ldots} \mathbb{Q} \xrightarrow{\text{adding } \pi , \text{e} , \text{sinn}, \ldots} \mathbb{R} \xrightarrow{\text{adopt } \sqrt{-1}} \mathbb{C} \)

  2-S forbidden \( \frac{1}{2} \) forb. completeness square roots algebraically

  (impossible to subtract larger number from a smaller number) \( x \in \mathbb{R} \) s.t. \( x^2 = 2 \)
* Notion of limits:

\[ \lim_{{x \to a}} f(x) = 6 \]

For each \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that if \( \lvert x - a \rvert < \delta \) then \( \lvert f(x) - 6 \rvert < \varepsilon \).

Notice that the order in \( \mathbb{R} \) was replaced by the notion of distance.

In short, one needs

- field structure
- notion of distance

on complex numbers to be able to establish calculus in a way similar to that of real variable.

Complex number: one adopts a new number \( \sqrt{-1} \).

but this is not a good notation because \( \sqrt{-1} \) actually represents two numbers.

\( \sqrt{-1} = \frac{\pm \text{i}}{2} \)

Roughly speaking, \( \mathbb{C} \) (complex numbers) consists of \( \mathbb{R} \) (real numbers) and \( \text{i} \) and all other numbers that can be generated through addition and multiplication.
Ex:

\[1 + 2i \quad \text{(better notation for } 1 + 2\sqrt{-1})\]

\[(2 + 3i)^2 = (2 + 3i)(2 + 3i) = 4 + 6i + 6i + 9i^2\]

\[= 4 + 12i + 9(-1)\]

\[= -5 + 12i\]

\[C = \{a + bi : a, b \in \mathbb{R}\}\]

Neutral element: \(0 + 0i \quad (= 0)\)

Unit element: \(1 + 0i \quad (= 1)\)

\(z = a + ib \quad \text{---- standard form}\)

\[\begin{array}{c}
\text{Re}(z) \\
\text{Im}(z)
\end{array}\]

\(z^{-1} = ?\), put \(z^{-1} = x + iy\)

Want: \((z + iy)(a + ib) = 1\)

\[\begin{cases}
a - b y = 1 \\
b x + a y = 0
\end{cases} \quad \text{system of 2 eqs., 2 unknowns.}\]

\[\begin{vmatrix}
1 & -b \\
0 & a
\end{vmatrix} = \frac{a}{a^2 + b^2}, \quad y = \ldots = \frac{-b}{a^2 + b^2}\]

\[x = \frac{a}{a^2 + b^2}, \quad y = \ldots = \frac{-b}{a^2 + b^2}\]

\[z^2 = (a + bi)^2 = a^2 - b^2 + 2abi\]

\[\bar{z} = a - bi \quad \text{complex conjugate of } z = a + bi\]

\[\frac{a + bi}{c + di} = \frac{(a + bi)(c - di)}{(c + di)(c - di)} = \frac{ac + bd + i(bc - cd)}{c^2 + d^2}\]

\[\begin{array}{c}
\text{nonstandard} \\
\text{form}
\end{array}
\]

\[= \frac{ac + bd}{c^2 + d^2} + i \frac{bc - cd}{c^2 + d^2}\]

\[\text{standard form}\]