We know that if a function \( f \) has an antiderivative on a region \( G \) and \( a \in G \), then \( \int_a^b \! f(z) \, dz \) only depends on the endpoints of \( Y \). More specifically,

\[
\int_a^b \! f(z) \, dz = F(b) - F(a)
\]

A useful consequence is the following:

Let \( f: G \subseteq \mathbb{C} \to \mathbb{C} \) and \( \gamma \) be a closed path (loop) contained in \( G \). Suppose \( f \) has an antiderivative on \( G \). Then

\[
\int \! f(z) \, dz = 0
\]

The property of "integral over a loop equal to zero" is a desirable feature of a complex function. If a function has an antiderivative, we know that it has this feature. However, it is sometimes hard to check if an antiderivative exists. It's easier to check if a derivative exists, for example using Cauchy–Riemann equations. Later in this lecture, we will state a result analogous to the above theorem, but for holomorphic functions (known as Cauchy–Goursat theorem).

- Geometric interpretation of complex integrals:

- \( f(z) = u(z) + iv(z) = (u(z), v(z)) \) can be thought as a vector field; to each point \( z \) is assigned a vector \( f(z) \).

- \( \overline{f(z)} = u(z) - iv(z) = (u(z), -v(z)) \) is the conjugate vector field, or \( \text{Poly} \) vector field of \( f \).
\[ \int_{\gamma} \mathbf{F} \cdot d\mathbf{s} = \int_{\gamma} (u, v) \cdot (dx, dy) = \mathbf{W} \left[ \mathbf{F}, \gamma \right] \quad \text{work done by } \mathbf{F} \text{ along } \gamma. \]

\[ \int_{\gamma} \mathbf{G} \cdot d\mathbf{x} = \int (u + iv)(dx + idy) = \int u \, dx - v \, dy + i \int v \, dy - u \, dx \]

\[ = \int (u, -v) \cdot (dx, dy) - (u, -v) \cdot (dy, -dx) \]

\[ = \int \mathbf{F} \cdot d\mathbf{s} + i \int \mathbf{G} \cdot d\mathbf{s} \]

\[ d\mathbf{s}^2 = (dy, -dx) \text{ is obtained by rotating } \]

\[ d\mathbf{s}^2 \text{ clockwise.} \]

Suppose \( \gamma \) is a simple closed curve, positively oriented; doesn't intersect itself.

GEV's theorem: \( \mathbf{W} \left[ \mathbf{F}, \gamma \right] = \int \text{curl} \mathbf{F} \, d\mathbf{A} = \int (\partial_y u - \partial_x v) \, d\mathbf{A} \)

Recall: \( \text{curl} \, (P, Q) = \partial_y Q - \partial_x P \)

\( \text{curl of a 2D-vector field is a scalar function} \)

\[ \text{curl} \, (P, Q, R) = \begin{vmatrix}
\mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\
\partial_x & \partial_y & \partial_z \\
P & Q & R
\end{vmatrix} \quad \text{curl of a 3D-vector field is a 3D-vector field.} \]
Divergence theorem:
\[ \int_{\Gamma} \nabla \cdot \mathbf{F} \, dA = \int_{\partial \Gamma} (\partial_{x}u - \partial_{y}v) \, dA \]

* Consequence: \( \nabla \cdot \mathbf{F} + i \mathbf{F} \cdot \mathbf{n} = \int_{\Gamma} \mathbf{F}(x) \, d\tau \)

Ex: Constant vector field \( \mathbf{g}(x,y) = (a_1, a_2) \), which can be considered as constant complex function
\[ \mathbf{g}(z) = a = a_1 + a_2 \]
\[ \mathbf{g}(z) = \bar{a} \], whose antiderivative is \( \bar{z} \).

For any loop \( \gamma \), \( \int_{\gamma} \mathbf{g}(x) \, dz = F[\mathbf{g}, \gamma] = 0 \) because \( \int_{\gamma} \bar{z} \, dz = 0 \).

Ex: Consider vector field \( \mathbf{f}(x,y) = (x^2, xy) \).

It can be viewed as complex function \( f(z) = x^2 + ixy \).

Polar vector field: \( \mathbf{F}(x) = z^2 - ixy \).

\( \int_{\gamma} \bar{z} \, dz = F[\mathbf{f}, \gamma] = \int_{\gamma} (z^2 - ixy) \, dz \)

Thm (Cauchy–Goursat): If \( f \) is holomorphic on the region \( G \) enclosed by a simple closed path \( \gamma \) and continuous on \( \partial G = \gamma \), then \( \int_{\gamma} f(z) \, dz = 0 \).

Why? If \( \gamma \) is a simple path, use Cauchy–Riemann eqns.
\[ \int_{\gamma} f(z) \, dz = \int_{\partial G} (-\partial_{x}u - \partial_{y}v) \, dA + i \int_{\partial G} (\partial_{x}u - \partial_{y}v) \, dA = 0 \]
If $\gamma$ is not a simple path, one decomposes it into simple paths.

\[
\oint f(z) \, dz = \oint_{\gamma_1} f(z) \, dz + \oint_{\gamma_2} f(z) \, dz + \oint_{\gamma_3} f(z) \, dz + \oint_{\gamma_4} f(z) \, dz
\]

once each of $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ is simple path.

\[
\oint_{\gamma_1} f(z) \, dz + \oint_{\gamma_2} f(z) \, dz + \oint_{\gamma_3} f(z) \, dz + \oint_{\gamma_4} f(z) \, dz = 0
\]

Ex: Function $f(z) = \frac{1}{z}$ is not holomorphic at $z=0$.

Cauchy-Goursat's thm not applicable

\[
\oint_{\gamma_2} \frac{1}{z} \, dz = 0
\]

because $\frac{1}{z}$ is holomorphic in the region enclosed by $\gamma_2$. 