Question: In what case can we write Laurent series?

Then let $f$ be holomorphic on an annulus $\mathbb{C}: r < |z| < R$, where $0 \leq r < R < \infty$. Then $f$ has a Laurent series representation about $0$:

$$f(z) = \ldots + a_{-2} z^{-2} + a_{-1} z^{-1} + a_0 + a_1 z + a_2 z^2 + \ldots$$

$$\mathbb{F}_2: \quad f(z) = \frac{1}{(z-1)(z-2i)}$$

Find the Laurent series representations of $f$ about $0$.

The singularities of $f(z)$ are $1$ and $2i$. They partition the complex plane into three zones:

1) $|z| < 1$
2) $1 < |z| < 2$
3) $|z| > 2$

By partial fraction, $f(z) = \frac{1}{1-2i} - \frac{1}{1-2i} = \frac{z}{z-2i}$.

In zone 2:

$$\frac{1}{z-1} = \frac{1}{z} \left( \frac{z}{z-1} \right) = \frac{1}{z} \left( 1 + \frac{1}{z} + \frac{1}{z^2} + \ldots \right) = \sum_{k=0}^{\infty} \frac{1}{z^k}$$

$$\frac{1}{z-2i} = \frac{-1}{2i} \left( \frac{z}{z-2i} \right) = \frac{-1}{2i} \sum_{k=0}^{\infty} \frac{z^k}{(2i)^k}$$

Substitute into $(\ast)$:

$$f(z) = \frac{1}{1-2i} \sum_{k=1}^{\infty} \frac{z^k}{z-1} + \frac{1}{4+2i} \sum_{k=0}^{\infty} \frac{z^k}{(2i)^k}$$
How many ways have we learned to compute integral over a loop?

1. Path parametrization:

\[ \oint_C f(z) \, dz = \int_0^1 f((y(t)) \gamma'(t) \, dt \]

- Advantage: no need to worry about the function being holomorphic, as long as it is continuous on the curve.
- Drawback: integrand can easily become too complicated in terms of \( t \).

2. Fundamental thm of Calc:

\[ \int_C f(z) \, dz = F(z) \bigg|_{y(a)}^{y(b)} = 0 \]

- Advantage: no need to worry about parametrization of the loop.
- Drawback: sometimes antiderivative DNE, or DNE in the region where the loop lies. Recall that for a function to have an antiderivative on a region, the function itself must be holomorphic there.

3. Cauchy-Goursat theorem:

\[ \oint_C f(z) \, dz = 0 \]

provided that \( f \) is holomorphic on the region enclosed by \( C \).

- Advantage: only need to check if the integrand is holomorphic, not much to be concerned about the parametrization of the loop.
4. Cauchy's Integral formula:

$$\oint_{\gamma} \frac{f(z)}{(z-a)^{n+1}} \, dz = \frac{2\pi i}{n!} \, f^{(n)}(a)$$

$\gamma$: simple, positively oriented

$f$ holomorphic inside $\gamma$

- Advantage: allows the integrand to have one singularity inside $\gamma$. Simple calculation.
- Drawback: doesn't allow integrand to be general fraction $\frac{f(z)}{g(z)}$

5. Cauchy's Residue theorem:

$$\oint_{\gamma} f(z) \, dz = 2\pi i \left( \text{Res}[f; z_1] + \cdots + \text{Res}[f; z_n] \right)$$

$\gamma$: simple loop, positively oriented

$z_1, z_2, \ldots, z_n$: isolated singularities of $f$ enclosed in $\gamma$.

- Advantage: allows $f$ to be quite general. Relatively simple calculation.
- Drawback: doesn't work if $f$ has non-isolated singularities inside $\gamma$.

**Question:** How to find the residues?

$$\text{Res}[f; z_0] = a_{-1} \quad \text{in the Laurent series} \quad f(z) = \sum_{k=-\infty}^{\infty} a_k (z-z_0)^k$$
If \( z_0 \) is removable, \( a_n = 0 \).

If \( z_0 \) is essential, compute \( a_n \) by trying to write Laurent series.

If \( z_0 \) is a pole of order \( n \): (let's take \( z_0 = 0 \) for simplicity)

\[
f(z) = a_n z^n + \ldots + a_1 z + a_0 + a_1 z + \ldots
\]

How do we know that \( z_0 \) is a pole of order \( n \)?

\[
\lim_{z \to z_0} z^n f(z) = a_n, \text{ which is neither 0 nor } \infty.
\]

**Example**

\[
f(z) = \frac{e^z - 1}{z \sin z}
\]

Is \( z = 0 \) a pole? What is its order?

We see that

\[
e^z - 1 = z + \frac{z^2}{2} + \ldots = z \left(1 + \frac{z}{2} + \ldots\right)
\]

\[
z \sin z = z \left(z - \frac{z^3}{6} + \ldots\right) = z \left(1 - \frac{z^2}{6} + \ldots\right)
\]

Then

\[
f(z) = \frac{z^{1/2} + \ldots}{z^2} \frac{1 + \frac{z}{2} + \ldots}{1 - \frac{z^2}{6} + \ldots}
\]

Thus, \( z^2 f(z) \to 1 \) as \( z \to 0 \).

\( z = 0 \) is a pole of order 2.

\[
\lim_{z \to 0} z^2 f(z) = a_n + a_{n+1} z + \ldots + a_{n+2} z^{n+1} + \ldots
\]

We see that \( a_n \) is the \( (n-1) \)st coefficient of the Taylor series of function \( g(z) = z^n f(z) \):

\[
a_{n-1} = \frac{1}{(n-1)!} g^{(n-1)}(0) = \frac{1}{(n-1)!} d^{n-1} \left(\frac{z^n f(z)}{z^n}\right)_{z=0}
\]

Another way to write it is:

\[
a_{n-1} = \frac{1}{(n-1)!} \lim_{z \to 0} \frac{d^{n-1}}{dz^{n-1}} \left((z-z_0)^n f(z)\right)
\]

Examples provided in worksheet.