Lecture 7 (4/15/2019)

\[ e^z = e^{x+iy} = e^x \cos(y) + ie^x \sin(y) \]

\[ \sin z = \frac{e^{iz} - e^{-iz}}{2i}, \quad \cos z = \frac{e^{iz} + e^{-iz}}{2} \]

\[ \cosh z = \frac{e^z + e^{-z}}{2}, \quad \sinh z = \frac{e^z - e^{-z}}{2} \]

(Start with Taylor series for real variable.

\[ \cos x, \sin x \]. Then formally plug \( ix \) for \( x \).

\[ \tan z = \frac{\sin z}{\cos z} = \frac{e^{iz} - e^{-iz}}{i} \frac{e^{iz} + e^{-iz}}{1} \quad \cot z = \frac{1}{\tan z} \]

How to define logarithm?

\[ \log z = w \quad \text{if} \quad e^w = z \]

Since exponential is not one-to-one, logarithm is a multi-valued function.

Solve for \( w \): write \( w = a + ib \)

\[ z = re^{i\theta} \]

\[ e^w = e^{a+ib} = re^{ib} \]

\[ \Rightarrow \begin{cases} e^a = r \\ b = \theta + 2k\pi \end{cases} \]

\[ \Rightarrow \begin{cases} a = \ln r \quad \text{(real logarithm)} \\ b = \theta + 2k\pi \end{cases} \]

The equation \( e^w = z \) has no solution \( w \in \mathbb{C} \) only if \( z = 0 \). Therefore, the range of \( e^z \) is \( C \setminus \{0\} \).

Thus,

\[ \log z = \ln r + i(\theta + 2k\pi) = \ln |z| + i \arg z \]

Visualize the real and imaginary part of logarithm:

\[ z \rightarrow \ln |z| \]

\[ \begin{cases} z = r \cos \theta \\ y = r \sin \theta \end{cases} \]

\[ \theta = \theta \]

\[ x \]

\[ \theta = \phi \]

\[ y \]
How to make logarithm a function?

Cut a branch of argument, for example the principal branch, where the negative real line $\mathbb{R}_{<0}$ is removed.

$\mathbb{C} \setminus \mathbb{R}_{\leq 0}$

$\text{the graph of function } z \mapsto \text{Arg}(z)$

$\log z = \ln |z| + i \text{Arg} z \sim \text{the principal branch of logarithm}$

**Ex.**

$log i = ? \quad \log i = ?$

$|i| = 1$

$\arg i = \frac{\pi}{2} + k2\pi$

$\mapsto \log i = \ln 1 + i \left( \frac{\pi}{2} + k2\pi \right) = i \left( \frac{\pi}{2} + k2\pi \right)$

$\quad \frac{\text{real}}{\text{imaginary}}$

$log i = \ln 1 + i \frac{\pi}{2} = i \frac{\pi}{2}$.

**Ex.**

$log e^i = \ln |e^i| + i \text{arg}(e^i) = \ln 1 + i (1 + k2\pi) = i(1 + k2\pi)$

$(e^i = e^\circ \cos 1)$

Another way: $i$ is a value of $\log e^i$. Thus, $\log e^i = \frac{1}{2} i + k2\pi i : k \in \mathbb{Z}$.

*Comments on branches of the logarithm:

The idea to introduce branches is to make sure that a multi-valued function is single-valued and “continuous” (i.e. without jump).

Logarithm is multi-valued because the argument is multi-valued.

Without graphing the surface of $\text{arg} z$, one can still tell that $\text{arg} z$ is not “continuous” on $\mathbb{C} \setminus \{0\}$, unless one “cuts” the complex plane.
Consider a loop passing \( z \) and enclosing 0 as in the picture. The argument increases in value as one moved from \( z \) along the curve. When one arrives at \( z \) again, the argument increases by \( 2\pi \). This is a jump (or "discontinuity").

To avoid such jump (or, to make argument a well-defined single-valued function), one needs to make sure that all curves enclosing 0 are excluded.

One way to do so is to cut the plane \( C \) by a curve \( C \) starting at 0 going to infinity and not intersecting itself.

(0 in this case is called a branch point.)

(\( C \) \( \cup \) \( C \) \( \cup \) \( C \) is called a branch cut.)

A branch cut cuts the surface of \( \arg z \) into infinitely many branches (think of floors in a building).
Power functions

\[ f(z) = \sqrt{z} \] is a multi-valued function

\[ z = r e^{i\theta} \]

Formally, \( f(z) = \sqrt{r} e^{i \theta/2} \)

As \( z \) moves along a curve (in the left picture), the argument increases. As \( z \) goes back to the original position, the arg. increases by \( 2\pi \). Thus, \( \sqrt{z} \) increases by \( \pi \), which flips the sign of \( f(z) = \sqrt{r} \).

A branch cut is needed. Choose, for example, \( C = \mathbb{R}_{\leq 0} \) (the negative real line).

Definition:

\[ z^a = e^{a \log z} \]

where \( \log z \) is the principal logarithm.

Implicitly, the definition uses the branch cut \( C = \mathbb{R}_{\leq 0} \).

Ex:

\[ i^{1/2} = e^{i \log i} = \exp \left( \frac{1}{2} i \frac{\pi}{2} \right) = \exp \left( i \frac{\pi}{4} \right) = \frac{\sqrt{2}}{2} + i \frac{\sqrt{2}}{2} \]

\[ (-1)^{1/3} = e^{i \log(-1)} = \exp \left( \frac{1}{3} i \pi \right) = \exp \left( i \frac{\pi}{3} \right) = \frac{1}{2} + i \frac{\sqrt{3}}{2} \]