* Recall: \( \log z = \ln |z| + i \arg z \)

\[ = \ln |z| + i (\arg z + k2\pi) \quad \text{(multi-valued function)} \]

\[ \log z = \ln |z| + i \underbrace{\arg z}_{\in (-\pi, \pi]} \quad \text{(principal branch of the \ logarithm)} \]

If \( z \in \mathbb{R} \) and \( z > 0 \), \( \log z = \ln z \).

The logarithm can be used to define various functions such as \( e^z \), \( \arcsin z \), \( \arccos z \), ... These are multi-valued functions.

Usually a branch cut is used to create a region on \( \mathbb{C} \)-plane such that the function is single-valued (so that one can do calculus on it).

Ex: \( \arg z \), \( \log z \) are well-defined functions on \( \mathbb{C} \setminus \mathbb{R}_{\leq 0} \).

There are infinitely many possible choices of branch cut for \( \arg z \).

However, each must start at 0 and go to infinity (to rule out...
any possible curve enclosing 0).

\[ f(z) = \text{Arg}z + \frac{\pi}{4} \in \left(-\frac{3\pi}{4}, \frac{5\pi}{4}\right) \]

branch cuts of Argz

**Ex:** What is the domain of \( f(z) = \log z \)?

\( C \setminus \mathbb{R}_<0 = C \setminus \{z \in \mathbb{C} : \text{Im}z = 0 \text{ and } z < 0\} \)

**Ex:** What is the domain of \( g(z) = \log(z^2) \)?

It consists of all complex numbers \( z \) except for those that satisfy \( z^2 \leq 0 \).

\[ \text{Arg}z = \frac{\pi}{2} \text{ or } -\frac{\pi}{2} \]

imply that \( \text{Arg}z^2 = \pi \).

Thus, the domain of \( z \) is everything but the imaginary axis: \( C \setminus i\mathbb{R} = C \setminus \{z : z = it, t \in \mathbb{R}\} \).

*Note:* \( \log z^2 \neq 2 \log z \) since the two functions have different domains.

More generally,

\[ \log(zw) \neq \log z + \log w \text{ in general} \]

This is because

\[ \text{Arg}(zw) \neq \text{Arg}z + \text{Arg}w. \]

(They are equal only in modulo \( 2\pi i \))

\[ \text{If } z \text{ and } w \text{ lie on the right half plane (that is, Re} z \text{ and Re} w \text{ are positive), then} \]

\[ \text{Arg}(zw) = \text{Arg}z + \text{Arg}w. \]
\[
\text{arg} (i(-1+i)) = \frac{\pi}{2} + \frac{3\pi}{4} + k2\pi = \frac{5\pi}{4} \pmod{2\pi}
\]

\[\Rightarrow \text{Arg} (i(-1+i)) = -\frac{3\pi}{4} \in (-\pi, \pi]\]

\[\text{Arg} i = \frac{\pi}{2}, \quad \text{Arg} (-1+i) = \frac{3\pi}{4}\]

Although each point \(z \in \mathbb{C}\{0\}\) has an argument (or more precisely, infinitely many arguments, each differing from another by a multiple of \(2\pi\)), it is impossible to define a single-valued, continuous argument function on \(\mathbb{C}\{0\}\).

There are two strategies to define a function from a multi-valued function:

1. If the function involves logarithm or argument, use principal branch of logarithm (Log) or of argument (Arg).
2. Identify branch points, introduce branch cuts, and pick a branch.

Ex: \[\log (z^2-1)\]

The cause of multi-value is the logarithm. A single-valued function can be defined by taking the principal branch of logarithm: \(f(z) = \log(z^2-1)\).

Note that the domain of \(f\) has to be carefully calculated:

\[\{z \in \mathbb{C} : z^2 - 1 \leq 0\}\]
Ex: $\sqrt{z^2 - 1}$, understood as multi-valued function $e^{\frac{1}{2} \log(z^2 - 1)}$.

One can define a single-valued function $\sqrt{z^2 - 1}$ by taking the principal branch of logarithm. Here is another method:

First, write $\sqrt{z^2 - 1} = \sqrt{(z-1)(z+1)}$

\[ \theta_1 = \text{Arg}(z-1) \in (-\pi, \pi] \]
\[ \theta_2 = \text{Arg}(z+1) \in (-\pi, \pi] \]

One writes formally that

\[ \text{Arg} \sqrt{z^2-1} = \frac{\theta_1 + \theta_2}{2} \in (-\pi, \pi] \]

As $z$ moves along a curve enclosing 1,

- $\theta_1$ increases by $2\pi$
- $\theta_2$ remains the same

at the time $z$ comes back to the original position.

\[ \frac{\theta_1 + \theta_2}{2} \] increases by $\pi$, which implies a jump in value.

Thus, 1 is a branch point. Similarly, -1 is also a branch point.

There are no other branch points.

Next, introduce branch cuts to forbid curves from enclosing -1 and 1.

There are many ways to do so. For example,
Next, define a branch (that is, to specify the range of $\theta_1$ and $\theta_2$).

\[ \theta_1 \in \left(-\frac{3\pi}{4}, \frac{5\pi}{4}\right) \] one branch
\[ \theta_2 \in \left(-\frac{3\pi}{2}, \frac{\pi}{2}\right) \]

\[ \theta_1 \in \left(\frac{5\pi}{4}, \frac{13\pi}{4}\right) \] another branch
\[ \theta_2 \in \left(\frac{\pi}{2}, \frac{5\pi}{2}\right) \]