Ex. Find domain of $f(z) = \sqrt{1-z^4}$ (principal branch being used).

We need to exclude any point $z \in \mathbb{C}$ such that $1-z^4 \leq 0$.

That is, $z^4 \geq 1$

In particular, $\text{Arg}(z^4) = 0$

We know that $\text{Arg}(z^4) = 4 \text{Arg}z \pmod{2\pi}$

$\Rightarrow 4 \text{Arg}z = 0 \pmod{2\pi}$

$\Rightarrow 4 \text{Arg}z = k \cdot 2\pi$

$\Rightarrow \text{Arg}z \in \left\{ \frac{\pi}{2}, \pi, \frac{3\pi}{2} \right\}$

$\Rightarrow z$ lies on the real and imaginary axes.

$z^4 \geq 1 \Rightarrow$ the modulus of $z$ is $\geq 1$.

\[ \text{Domain } = \mathbb{C} \setminus \{z : z = a \text{ with } a \in \mathbb{R}, |a| \geq 1 \} \]

or $z = ia$ with $a \in \mathbb{R}, |a| \geq 1$.

One sees that the 4 points $\pm 1, \pm i$ are branch points of the multi-valued function $\sqrt{1-z^4}$. Indeed, write

$\sqrt{1-z^4} = \sqrt{(1-z)(z+1)(z-i)(z+i)}$

$\theta_1 = \text{Arg}(1-z)$

$\theta_2 = \text{Arg}(z+1)$

$\theta_3 = \text{Arg}(z-i)$

$\theta_4 = \text{Arg}(z+i)$
As \( z \) moves along a small closed curve enclosing 1,

\[ \theta_1 \text{ increases by } 2\pi, \]

\[ \theta_2 \text{ returns to its original value,} \]

\[ \theta_3 \text{ and } \theta_4 \text{ also return.} \]

The quantity \( \frac{\theta_1 + \theta_2 + \theta_3 + \theta_4}{2} \) increases by \( \pi \), which means \( w(z) \) does not return to its original position. The branch cuts (the red lines above) forbid any closed curve in the domain to enclose any of the branch points \( \pm i, \pm 1 \).

**Ex:** find the domain of \( f(z) = \sqrt{1+z} \sqrt{1-z} \) (principal branch).

Remove all \( z \)'s such that \( 1+z \leq 0 \) or \( 1-z \leq 0 \).

For the first part, \( z \leq -1 \).

For the second part, \( i-z = a \leq 0 \)

\[ \Rightarrow \quad z = c-a \quad (a \leq 0) \]

\[ \Rightarrow \quad z = c+b \quad (b \geq 0) \]

This is equation of a line \( y=1 \)

[Diagram showing the domain with lines and points labeled i and -1]
Ex: find a branch \( f(z) \) of \( z^{1/3} \) such that \( f(1) = \text{cis} \frac{i\pi}{3} \).

If the principal branch of logarithm were chosen, \( f(1) \) would be \( z_1 = 1 \).

\[
f(z) = e^{\frac{i}{3} \log z} = e^{\frac{1}{3} (\ln |z| + i \text{Arg} z + 2k\pi)} = |z|^{1/3} e^{i \text{Arg} z + i 2k\pi / 3} \]

\((k = 0, 1, 2)\)

Plug \( z = 1 \): \( f(1) = e^{(0 + i 2k\pi)/3} = e^{i \frac{2k\pi}{3}} \)

It was given that \( f(1) = \text{cis} \frac{i\pi}{3} \). Thus, \( k = 1 \).

Here is the definition of \( f(z) \):

\[
f(z) = |z|^{1/3} e^{i \text{Arg} z + i 2\pi / 3} = |z|^{1/3} e^{i \frac{2\pi i}{3}}
\]

(principal branch)

In other words, \( f(z) \) differs from the principal branch of \( z^{1/3} \) by a rotation of \( \frac{2\pi}{3} \) counter-clockwise.

*Recall: \( z^a = e^{a \log z} \)

Special case: \( a \in \mathbb{R} \). In this case,

\[
z^a = e^{a (\ln |z| + i \text{Arg} z)} = e^{a \ln |z| + e^{ia \text{Arg} z}} = |z|^a \text{cis} (a \text{Arg} z)
\]

Rule of thumb: \( z^a \) \((a \in \mathbb{R}) \) 
- raise modulus to power \( a \),
- multiply principal arg by \( a \).

(similar to positive integer power)
* Properties:

1) \[ z^{a+b} = z^a z^b \]

- using the same branch of logarithm

Why?

\[ \text{LHS} = e^{(a+b) \log z} = e^{a \log z + b \log z} = e^{a \log z} e^{b \log z} = z^a z^b \]

2) \[ \frac{1}{z^a} = z^{-a} \]

3) \[ z^{-a} = \frac{1}{z^a} \]

4) \[ (z^a)^n = z^{an} \text{ if } n \text{ is integer} \]

This equality is not necessarily true if \( n \) is complex.

* Caution:

\[ (zw)^n \neq z^n w^n \text{ in general} \]

It comes from the fact that \( \log(zw) \neq \log z + \log w \), which comes from the fact that

\[ \text{Arg}(zw) \neq \text{Arg} z + \text{Arg} w \]