1. Consider function \( f(z) = \frac{\log(z+5)}{\sin z} \).

(a) Determine all singular point(s) of \( f \) enclosed in the circle \( C_4(0) \). Are they isolated singularities?

Note that the roots of \( \sin z = 0 \) always lie on the real axis.

\( \sin z = 0 \) when \( z = k\pi \) where \( k \in \mathbb{Z} \). The only roots of \( \sin z \) enclosed by \( \Gamma = C_4(0) \) are \( z = 0, \pm \pi \).

Function \( f \) also has nonisolated singularities (on the line \( \mathbb{R}_{\leq -5} \)), but these points lie outside of \( \Gamma \).

Conclusion: \( 0, \pm \pi \) are the only singularities of \( f \) inside \( \Gamma \). They are isolated singularities.

(b) Which kind of isolated singularity are they (removable, pole, essential)? If they are poles, determine their orders.

When \( z \) is near to 0, \( \log(z+5) \) is near to \( \ln 5 \), and

\[
\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots = z \left( 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \ldots \right)
\]

Thus,

\[
f(z) \approx \frac{\log 5}{z \left( 1 - \frac{z^2}{2!} + \ldots \right)} = z^{-1} \frac{\log 5}{1 - \frac{z^2}{2!} + \ldots}
\]

\( \neq 0 \) when \( z = 0 \)

we guess that 0 is a pole of order 1. To verify our guess, we compute

\[
\lim_{z \to 0} z f(z) = \lim_{z \to 0} \frac{z \log(z+5)}{\sin z} = \ln 5 \lim_{z \to 0} \frac{1}{\sin \frac{z}{2}}
\]

\( \lim_{z \to 0} \frac{1}{\sin \frac{z}{2}} = \frac{1}{2} \ln 5 \neq 0, \neq \infty
\]

Similarly, \( \pm \pi \) are also poles of order 1 because

\[
\lim_{z \to \pm \pi} (z-\pm \pi) f(z) = \lim_{z \to \pm \pi} \frac{\log(z+5)}{\sin z} = \ln(5 \pm \pi) \lim_{z \to 0} \frac{z-\pi}{\sin z} = \pm \ln(5 \pm \pi)
\]

\( \lim_{z \to \pm \pi} (z-\pm \pi) f(z) = \ldots \)
(c) Compute the residue of $f$ at each of these singularities.

\begin{align*}
\text{use the formula } \quad \text{Res } [f, z_0] = a_{-1} &= \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} [(z-z_0)^n f(z)] \\
\text{Here } n &= 1 \text{ for } z_0 = 0, \pm \pi i.
\end{align*}

\begin{align*}
\text{Res } [f, 0] &= \frac{1}{0!} \lim_{z \to 0} \frac{d^0}{dz^0} [(z-0)^1 f(z)] = \lim_{z \to 0} f(z) = \ln 5 \text{ (as computed above)} \\
\text{Res } [f, \pi i] &= \cdots = \lim_{z \to \pi i} (z-\pi i)f(z) = -\ln (5+\pi) \text{ (as computed above)} \\
\text{Res } [f, -\pi i] &= \cdots = \lim_{z \to -\pi i} (z+\pi i)f(z) = -\ln (5-\pi)
\end{align*}

(d) Evaluate the integral $\gamma f(z)dz$ where $\gamma$ is the circle $C_4(0)$ oriented counterclockwise.

By Cauchy's Residue theorem,

\begin{align*}
\int_{\gamma} f(z)dz &= 2\pi i \left( \text{Res } [f, 0] + \text{Res } [f, \pi i] + \text{Res } [f, -\pi i] \right) \\
&= 2\pi i \left( \ln 5 - \ln (5+\pi) - \ln (5-\pi) \right) \\
&= 2\pi i \ln \frac{5}{25-\pi^2}
\end{align*}
2. Compute $\int_{\gamma} \frac{z+1}{(z-\frac{\pi}{2})^2 \sin z} \, dz$ where $\gamma$ is the circle $C_2(0)$ oriented counterclockwise.

The only singularities of $f(z) = \frac{z+1}{(z-\frac{\pi}{2})^2 \sin z}$ that lie in $\gamma$ are $z=0$ and $z=\frac{\pi}{2}$.

We see that $f(z) = \frac{z+1}{(z-\frac{\pi}{2})^2 \sin z} = \frac{z+1}{(z-\frac{\pi}{2})^2 (1-\frac{z^2}{6} + \ldots)} \neq 0$ when $z = \frac{\pi}{2}$

and

$$f(z) = \frac{z+1}{(z-\frac{\pi}{2})^2 (\frac{z^2}{6} + \ldots)} = \frac{z+1}{(z-\frac{\pi}{2})^2 (1-\frac{z^2}{6} + \ldots)} \neq 0 \text{ when } z = 0$$

Thus, we guess that $z = \frac{\pi}{2}$ is a pole of order 2, $z = 0$ is a pole of order 1.

Verify:

$$\lim_{z \to \frac{\pi}{2}} (z-\frac{\pi}{2}) f(z) = \lim_{z \to \frac{\pi}{2}} \frac{z+1}{\sin z} = \frac{\pi/2 + 1}{1} \neq 0, \neq \infty$$

$$\lim_{z \to 0} z f(z) = \lim_{z \to 0} \frac{(z+1)z}{(z-\frac{\pi}{2})^2 \sin z} = \frac{1}{(\frac{\pi}{2})^2} \lim_{z \to 0} \frac{z+1}{\sin z} \frac{1}{\sin \frac{\pi}{2}} \neq 0, \neq \infty$$

Next,

$$\text{Res } [f, \frac{\pi}{2}] = \frac{1}{1!} \lim_{z \to \frac{\pi}{2}} \frac{d}{dz} \left[ (z-\frac{\pi}{2}) f(z) \right] = \lim_{z \to \frac{\pi}{2}} \frac{d}{dz} \left( \frac{z+1}{\sin z} \right) \neq 0$$

$$\text{Res } [f, 0] = \frac{1}{0!} \lim_{z \to 0} z f(z) = \frac{4}{\pi^2} \text{ (as computed above).}$$

By Cauchy's Residue theorem,

$$\int f(z) \, dz = 2\pi i \left( \text{Res } [f, \frac{\pi}{2}] + \text{Res } [f, 0] \right) = 2\pi i \left( 1 + \frac{4}{\pi^2} \right)$$
3. Compute \(\gamma z^2 \sin\left(\frac{1}{z}\right)dz\) where \(\gamma\) is the boundary of square with vertices at \(\pm 1 \pm i\) negatively oriented.

The only possible singularity of \(f(z) = z^2 \sin\left(\frac{1}{z}\right)\) is \(z=0\). The function is holomorphic everywhere else. Thus, \(f(z)\) can be written as a Laurent series around \(0\):

Let's try to write a Laurent series of \(f(z)\):

\[
\sin w = w - \frac{w^3}{3!} + \frac{w^5}{5!} - \ldots
\]

Now substitute \(w = \frac{1}{z} = z^{-1}\):

\[
\sin \frac{1}{z} = \frac{1}{z} - \frac{z^{-3}}{3!} + \frac{z^{-5}}{5!} - \ldots
\]

Thus,

\[
f(z) = z^2 \sin \frac{1}{z} = z - \frac{z^{-1}}{3!} + \frac{z^{-3}}{5!} - \ldots
\]

This is the Laurent series of \(f(z)\) around \(0\). (And we see that \(0\) is an essential singularity)

\[
\text{Res}[f; 0] = \text{coefficient of } z^{-1} = -\frac{1}{6}
\]

By Cauchy's Residue theorem,

\[
\oint_{\gamma} f(z)dz = -2\pi i \text{ Res}[f; 0] = -2\pi i \left(-\frac{1}{6}\right) = \frac{\pi i}{3}
\]

positively oriented loop