Exercise 1. Find all complex roots of the following cubic equation. Write them in standard form $z = a + ib$ where $a$ and $b$ are numerical values (round to 4 digits after decimal point).

(a) $z^3 + 3z + 1 = 0$

Answer: From the Cardano’s formula (see also the lecture note for roots of cubic polynomial), we split $z$ into the sum of $u, v$: $z = u + v$. Then note that

$$\Delta = q^2 + \frac{4p^3}{27} = 1 + \frac{4}{27} = 5$$

and so

$$u = \left[\frac{\sqrt{5} - 1}{2}\right]^{1/3}$$

The third roots of $\frac{\sqrt{5} - 1}{2}$ are

$$\Gamma_1 := \left\{ \left(\frac{\sqrt{5} - 1}{2}\right)^{1/3} e^{i\left(\frac{2k\pi}{3}\right)} : k = 0, 1, 2 \right\}$$

Similarly,

$$v = \left[\frac{-\sqrt{5} - 1}{2}\right]^{1/3}$$

and the third roots of $\frac{-\sqrt{5} - 1}{2}$ are

$$\Gamma_2 := \left\{ \left(\frac{\sqrt{5} + 1}{2}\right)^{1/3} e^{i\left(-\frac{\pi + 2m\pi}{3}\right)} : m = 0, 1, 2 \right\}$$

Let

$$u_1 = \left(\frac{\sqrt{5} - 1}{2}\right)^{1/3}, \quad u_2 = \left(\frac{\sqrt{5} - 1}{2}\right)^{1/3} e^{i\frac{2\pi}{3}}, \quad u_3 = \left(\frac{\sqrt{5} - 1}{2}\right)^{1/3} e^{i\frac{4\pi}{3}}$$
and
\[ v_1 = \left( \frac{\sqrt{5} + 1}{2} \right)^{1/3} e^{-i\pi/3}, \quad v_2 = \left( \frac{\sqrt{5} + 1}{2} \right)^{1/3} e^{i\pi/3}, \quad v_3 = \left( \frac{\sqrt{5} + 1}{2} \right)^{1/3} e^{i\pi} \]

We choose the pairs of \( u \) and \( v \) such that \( uv = -p/3 = -1 \). A quick way to do this is to choose the pairs of \( u \) and \( v \) such that \( \arg u + \arg v = \arg(-1) = \pi \) (in modulo \( 2\pi \)). These pairs are \((u_1, v_3), (u_2, v_2), (u_3, v_1)\). Therefore, the three complex roots of the given equation are

\[
\begin{align*}
z_1 &= u_1 + v_3 \approx -0.3222 \\
z_2 &= u_2 + v_2 \approx 0.1611 + i1.7544 \\
z_3 &= u_3 + v_1 \approx 0.1611 - i1.7544
\end{align*}
\]

You can double-check your results with Mathematica by the command \text{NSolve}:

\text{NSolve}[z^3 + 3z + 1 == 0, z]

(b) \(2z^3 - 6z^2 + 2z + 1 = 0\)

\textbf{Answer:} Set \( a = 2, \ b = -6, \ c = 2, \ d = 1\).

Divide by the leading coefficient:

\[
z^3 - 3z^2 + z + \frac{1}{2} = 0
\]

and then use the change of variables: \( x := z + \frac{b}{3a} = z - 1 \), or equivalently, \( z = x + 1 \). Substitute this into the equation (1):

\[
(x + 1)^3 - 3(x + 1)^2 + (x + 1) + \frac{1}{2} = 0
\]

multiply them out and simplify to obtain

\[
x^3 - 2x - \frac{1}{2} = 0
\]

Use the same argument as we did in part (a), we find that

\[
\Delta = -\frac{101}{108} = \frac{101}{108}i^2 \implies \sqrt{\Delta} = \pm \frac{\sqrt{101}}{108}i
\]

so

\[
u = \left( \frac{1}{4} + i\sqrt{\frac{101}{432}} \right)^{1/3}, \quad v = \left( \frac{1}{4} - i\sqrt{\frac{101}{432}} \right)^{1/3}
\]
we get

\[ u_k = \left( \frac{2\sqrt{6}}{9} \right)^{1/3} e^{i(\theta_1+2k\pi)/3}, \quad k = 0, 1, 2 \]

where \( \theta_1 = \arctan(\sqrt{101/27}) \)

\[ v_k = \left( \frac{2\sqrt{6}}{9} \right)^{1/3} e^{i(\theta_2+2k\pi)/3}, \quad k = 0, 1, 2 \]

where \( \theta_2 = \arctan(-\sqrt{101/27}) = -\theta_1. \)

We choose the pairs of \( u \) and \( v \) such that \( uv = -p/3 = 2/3 \). A quick way to do this is to choose the pairs of \( u \) and \( v \) such that \( \arg u + \arg v = \arg(2/3) = 0 \) (in modulo \( 2\pi \)). These pairs are \((u_1, v_3), (u_2, v_2), (u_3, v_1)\). Therefore, the three complex roots of the given equation are \((u_1, v_1), (u_2, v_2), (u_3, v_3)\). Therefore, the three values of \( x \) are

\[ x_1 = u_1 + v_1 \approx 1.5257 \]

\[ x_2 = u_2 + v_2 \approx -1.2670 \]

\[ x_3 = u_3 + v_3 \approx -0.2587 \]

Therefore, the three roots of the given equation are

\[ z_1 = u_1 + v_1 \approx 2.5257 \]

\[ z_2 = u_2 + v_2 \approx -0.2670 \]

\[ z_3 = u_3 + v_3 \approx 0.7413 \]

Exercise 2. Express the following complex numbers in either standard form or polar form.

(a) \( e^{e^{1+2i}} = e^{e^{i(\cos(2) + i \sin(2))}} = e^{e \cos(2) e^{i \sin(2)}} = e^{e \cos(2) \cos(e \sin(2)) + ie^{e \cos(2) \sin(e \sin(2))}} \)

(b) \( e^{2i} = e^{\sqrt{5}} \)

(c) \( \sin(-1 + i) = \frac{e^{(-1+i)} - e^{-(-1+i)}}{2i} = \frac{e^{-1}e^{-i} - ee^{i}}{2i} = \frac{-e^{-1}(1 - i \sin 1) - e(1 + i \sin 1)}{2i} \)

\( = \frac{(e^{1} - e) \cos 1 + i(-e^{1} - e) \sin 1}{2i} = \left( \frac{-e + e^{-1}}{2} \right) \sin 1 + i \left( \frac{e - e^{-1}}{2} \right) \cos 1 \)

(d) \( \tan(i) = \frac{e^{i} - e^{-i}}{i(e^{i} + e^{-i})} = \frac{e - e^{-1}}{i(e + e^{-1})} \)
Exercise 3. Recall de Moivre’s formula:

\[(\cos x + i \sin x)^n = \cos(nx) + i \sin(nx)\]

for any real number \(x\) and integer number \(n\). Apply this formula to express \(\cos(5x)\) and \(\sin(5x)\) in terms of \(\cos x\) and \(\sin x\)

Proof. Note that

\[
(\cos x - i \sin x)^5 = (\cos(-x) + i \sin(-x))^5 = \cos(-5x) + i \sin(-5x)
\]

\[
= \cos(5x) - i \sin(5x)
\]

(2)

On the other hand, de Moivre’s formula gives us:

\[
(\cos x + i \sin x)^5 = \cos(5x) + i \sin(5x)
\]

(3)

Combine (3) and (3), we have

\[
\cos(5x) = \frac{(\cos x + i \sin x)^5 + (\cos x - i \sin x)^5}{2}
\]

and

\[
\sin(5x) = \frac{(\cos x + i \sin x)^5 - (\cos x - i \sin x)^5}{2i}
\]

Exercise 4. Recall that the sine and cosine functions are defined in terms of the exponential function. Use the identity \(e^{u+v} = e^u e^v\) for any \(u, v \in \mathbb{C}\) to prove the following identity:

\[
\sin(z + w) = \sin z \cos w + \cos z \sin w \quad \forall z, w \in \mathbb{C}
\]

Proof. Recall that

\[
\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \quad \cos z = \frac{e^{iz} + e^{-iz}}{2}
\]

So

\[
\sin z \cos w + \cos z \sin w = \frac{e^{iz} - e^{-iz}}{2i} \frac{e^{iw} + e^{-iw}}{2} + \frac{e^{iz} + e^{-iz}}{2} \frac{e^{iw} - e^{-iw}}{2i}
\]

\[
= \frac{e^{i(z+w)} + e^{i(z-w)} - e^{-i(z-w)} - e^{-i(z+w)}}{4i}
\]

\[
= \frac{2e^{i(z+w)} - 2e^{-i(z+w)}}{4i} = \frac{e^{i(z+w)} - e^{-i(z+w)}}{2i} = \sin(z + w)
\]
Exercise 5. For $z, w \in \mathbb{C}$, show the following identities. We let $z = a + ib$ and $w = c + id$ in the following.

(a) $\overline{z + w} = \bar{z} + \bar{w}$:

Proof. Since $z + w = (a + c) + i(b + d)$ so,

$$\overline{z + w} = (a + c) - i(b + d) = (a - ib) + (c - id) = \bar{z} + \bar{w}$$

(b) $\overline{zw} = \bar{z}\bar{w}$:

Proof. $zw = (ac - bd) + i(ad + bc) \implies \overline{zw} = (ac - bd) - i(ad + bc)$ and so

$$\bar{z}\bar{w} = (a - ib)(c - id) = (ac - bd) - i(ad + bc) = \overline{zw}$$

(c) $|zw| = |z||w|$

Proof.

$$|zw| = \sqrt{(ac - bd)^2 + (ad + bc)^2} = \sqrt{(a^2c^2 - 2abcd + b^2d^2) + (a^2d^2 + 2abcd + b^2c^2)}$$

$$= \sqrt{a^2(c^2 + d^2) + b^2(c^2 + d^2)}$$

$$= \sqrt{a^2 + b^2}\sqrt{c^2 + d^2}$$

$$= |z||w|$$

(d) $\overline{\left(\frac{z}{w}\right)} = \frac{\bar{z}}{\bar{w}}$ where $w \neq 0$:

Proof. Note that

$$\frac{z}{w} = \frac{\bar{z}\bar{w}}{\bar{w}\bar{w}} = \frac{\bar{z}\bar{w}}{|w|^2}$$

so

$$\overline{\left(\frac{z}{w}\right)} = \overline{\left(\frac{\bar{z}\bar{w}}{|w|^2}\right)} = \frac{\bar{z}\bar{w}}{|w|^2} = \frac{\bar{z}}{\bar{w}}$$

(e) $|z^n| = |z|^n$: 

Proof. Note that this identity holds if \( z = 0 \).

Assume \( z \neq 0 \) and \( z = re^{i\theta} \) for some \( r > 0 \) and \( \theta \in [-\pi, \pi) \). Then

\[
|z^n| = |(re^{i\theta})^n| = |r^n e^{i\theta}| \quad \text{(de Moivre’s formula)}
\]

\[
= r^n |e^{i\theta}| = r^n = |z|^n
\]

Exercise 6. (a) Use Mathematica to plot the image of the line \( y = -1 \) under the function \( f(z) = z^2 \):

(b) Use Mathematica to plot the image of the unit circle \( x^2 + y^2 = 1 \) under the function \( f(z) = z^2 \):
(c) Re-do part (a) and (b) for $f(z) = z^3$:

Image of the line $y = -1$ under $z^3$:

Image of the unit circle $x^2 + y^2 = 1$ under $z^3$:
(d) Re-do part (a) and (b) for $f(z) = 1/z$:

Image of the line $y = -1$ under $1/z$:

```
\text{Out[5]} = \text{ParametricPlot}[\text{ReIm}[\text{f}[-\text{t} - \text{I}]], \{\text{t}, -9, 8\}, \text{AspectRatio} \to \text{Automatic}, \text{AxesOrigin} \to \{0, 0\}]
```

Image of the unit circle $x^2 + y^2 = 1$ under $1/z$:

```
\text{Out[8]} = \text{ParametricPlot}[\text{ReIm}[\text{f}[-\text{t} + \text{I}]], \{\text{t}, 0, 2\pi\}, \text{AspectRatio} \to \text{Automatic}, \text{AxesOrigin} \to \{0, 0\}]
```
\textbf{In[1]} = f[z_] := 1/z

\textbf{In[2]} = ParametricPlot[ReIm[f[Cos[t] + Sin[t] \cdot t]], \{t, 0, 2 \pi\}, AspectRatio \to \text{Automatic}, AxesOrigin \to \{0, 0\}]