Exercise 1. Write the following complex numbers in either standard or polar form.

(a) \((1 + i\sqrt{3})^{i+1}\)

Answer: Since

\[
1 + i\sqrt{3} = 2 \left( \frac{1}{2} + i \frac{\sqrt{3}}{2} \right) = 2 e^{i \left( \frac{\pi}{3} + k2\pi \right)}
\]

Thus,

\[
(1 + i\sqrt{3})^{i+1} = e^{(i+1) \log(1+i\sqrt{3})} = e^{(1+i)(\ln 2+i(\frac{\pi}{3}+k2\pi))}
\]

\[
= e^{(\ln 2 - \frac{\pi}{3} - k2\pi) + i(\ln 2 + \frac{\pi}{3} + k2\pi)}, \quad k \in \mathbb{Z}
\]

(b) \((-1)^{\sqrt{2}}\)

Answer:

\[
(-1)^{\sqrt{2}} = e^{\sqrt{2} \log(-1)} = e^{\sqrt{2}i(-\pi + k2\pi)}
\]

\[
= e^{i(-\sqrt{2} + 2\sqrt{2}k)\pi}, \quad k \in \mathbb{Z}
\]

(c) \((-1 - i)^{1/4}\)

Answer:

\[
(-1 - i)^{1/4} = e^{\frac{1}{4} \log(-1-i)} = e^{\frac{1}{4}(\ln \sqrt{2} + i(-\frac{3\pi}{4} + k2\pi))}
\]

\[
= 2^{\frac{1}{4}} \left( \cos\left( -\frac{3\pi}{16} + \frac{k\pi}{2} \right) + i \sin\left( -\frac{3\pi}{16} + \frac{k\pi}{2} \right) \right), \quad k \in \mathbb{Z}
\]

(d) \(\log\left( e^{i\frac{10\pi}{7}} \right)\)
Answer:
\[ \log \left( e^{i \frac{19}{4} \pi} \right) = \log \left( e^{i(\frac{19}{4} \pi + k2\pi)} \right) \]
\[ = i \left( \frac{19}{4} \pi + k2\pi \right), \quad k \in \mathbb{Z} \]

(e) \( e^{i\sqrt{2}} \)

Answer: Note that \( e^{i\sqrt{2}} \) is not computed as \( e^{i\sqrt{2} \log e} \). We know that
\[ e^{i\sqrt{2}} = \cos(\sqrt{2}) + i \sin(\sqrt{2}) \]

Exercise 2. Find all complex numbers satisfying the following equation. Write your results in either standard form or polar form.

(a) \( e^z = -1 \).

Answer: Write \( z = a + ib \), then
\[ e^z = e^{a+ib} = e^a e^{ib} = -1 = e^{i(2k+1)\pi}. \]
Therefore,
\[ z = i(2k+1)\pi \]
Alternatively, one can directly write \( z = \log(-1) = \ln | -1 | + i \text{Arg}(-1) = i(\pi + 2k\pi) \).

(b) \( 2^z = 4 \).

Proof. There are two ways to solve this problem.

Method 1: Because \( 2^z = e^{z \log 2} \), we get \( e^{z \log 2} = 4 \). This equation simply means \( z \log 2 = \log 4 \). Thus,
\[ z = \frac{\log 4}{\log 2} = \frac{\ln 4 + ik2\pi}{\ln 2 + il2\pi} = \frac{(\ln 4 + ik2\pi)(\ln 2 - il2\pi)}{(\ln 2 + il2\pi)(\ln 2 - il2\pi)} = \frac{(\ln 4)(\ln 2) + k\ln 2\pi + i2\pi(k \ln 2 - l \ln 4)}{(\ln 2)^2 + (l2\pi)^2} = \frac{2\pi(k \ln 2 - l \ln 4)}{(\ln 2)^2 + (l2\pi)^2}.

Method 2: Write \( z = a + ib \), then
\[ 2^z = 2^{a+ib} = e^{(a+ib)(\log 2)} = e^{(a+ib)(\ln 2 + ik2\pi)} = e^{(a \ln 2 - bk2\pi)} e^{i(b \ln 2 + ak2\pi)} \]
Thus, $2^z = 4$ is equivalent to
\[ e^{(a \ln 2 - bk2\pi)} e^{i(b \ln 2 + ak2\pi)} = 4 = 4 e^{i2m\pi}, \quad m \in \mathbb{Z} \]

So,
\[
\begin{cases}
a \ln 2 - bk2\pi = 2 \ln 2 \\
b \ln 2 + ak2\pi = 2m\pi, \quad m \in \mathbb{Z}
\end{cases}
\tag{1}
\]

This is a linear system of two unknowns ($a$ and $b$). Solving this system of equations, we obtain
\[
a = \left( \ln 4 + \frac{4km\pi^2}{\ln 2} \right) / \left( \ln 2 - \frac{4k^2\pi^2}{\ln 2} \right)
\]
and
\[
b = \frac{1}{\ln 2} \left[ 2m\pi - 2k\pi \left( \ln 4 + \frac{4km\pi^2}{\ln 2} \right) / \left( \ln 2 - \frac{4k^2\pi^2}{\ln 2} \right) \right]
\]
for $m, k \in \mathbb{Z}$.

(c) $\sin z = 1 - 2i$
\[
z = \arcsin (1 - 2i) = \frac{1}{i} \log(i(1 - 2i + \sqrt{1-(1-2i)^2})) = -i \log(2 + i + \sqrt{4 + 4i}).
\]
We have
\[
\sqrt{4 + 4i} = \sqrt{\sqrt{32}e^{i\pi/4}} = \pm \sqrt{32}e^{i\pi/8}.
\]
Thus,
\[
z = -i \log(2 + i \pm \sqrt{32}e^{i\pi/8})
\]
If the plus sign is taken then
\[
z = -i \log(2 + i + \sqrt{32}e^{i\pi/8}) = -i \log \left( 2 + \sqrt{32} \cos \frac{\pi}{8} + i \left( 1 + \sqrt{32} \sin \frac{\pi}{8} \right) \right)
\]
\[
= -i(\ln r + i(\theta + k2\pi))
\]
\[
= \theta + k2\pi - i \ln r
\]
where $r \approx 4.6116$ and $\theta \approx 0.4271$. Thus,
\[
z \approx 0.4271 + k2\pi - i1.5286
\]
If the minus sign is taken then
\[
z = -i \log(2 + i - \sqrt{32}e^{i\pi/8}) = -i \log \left( 2 - \sqrt{32} \cos \frac{\pi}{8} + i \left( 1 - \sqrt{32} \sin \frac{\pi}{8} \right) \right)
\]
\[
= -i(\ln s + i(\gamma + m2\pi))
\]
\[
= \gamma + k2\pi - i \ln s
\]
where $s \approx 0.2168$ and $\gamma \approx 2.7145$. Thus,
\[
z \approx -1.52857 + m2\pi - i2.7145
\]
In conclusion, the solutions of the equation \( \sin z = 1 - 2i \) are

\[
z \in \{0.4271 + k2\pi - i1.5286 : k \in \mathbb{Z}\} \cup \{-1.52857 + m2\pi - i2.7145 : m \in \mathbb{Z}\}.
\]

(d) \( \cos z = 1/2 \)

\textit{Answer:} Note that

\[
\cos z = \frac{e^{iz} + e^{-iz}}{2} = \frac{1}{2} \iff e^{iz} + e^{-iz} = 1
\]

\[
\iff e^{2i\pi} - e^{i\pi} + 1 = 0
\]

Again, solving this equation by quadratic formula,

\[
e^{iz} = \frac{1 \pm \sqrt{3}}{2}
\]

We get \( iz = \log \left( \frac{1 \pm \sqrt{3}}{2} \right) \), or equivalently

\[
z = -i \log \left( \frac{1 \pm i\sqrt{3}}{2} \right)
\]

If the plus sign is taken then

\[
z = -i \log \left( \frac{1 + i\sqrt{3}}{2} \right) = -i \log \left( e^{i\frac{\pi}{3}} \right) = -ii \left( \frac{\pi}{3} + 2k\pi \right) = \frac{\pi}{3} + 2k\pi
\]

If the minus sign is taken then

\[
z = -i \log \left( \frac{1 - i\sqrt{3}}{2} \right) = -i \log \left( e^{-i\frac{\pi}{3}} \right) = -ii \left( -\frac{\pi}{3} + 2m\pi \right) = -\frac{\pi}{3} + 2m\pi
\]

In conclusion, the set all solutions \( z \) is

\[
z \in \left\{ \pm \frac{\pi}{3} + 2k\pi : k \in \mathbb{Z} \right\}
\]

Exercise 3. Decide if each of the following statements is true or false. If it is true, prove it. If it is false, give a counterexample.

(a) \( \Log(z) + \Log(w) = \Log(zw), \forall z, w \in \mathbb{C}, z, w \neq 0 \).

\textit{Proof.} False.

Let \( z = w = -1 \). Then \( \Log(z) = \Log(w) = \ln(1) + i\pi = i\pi \)
so
\[ \log(z) + \log(w) = i2\pi \]

but
\[ \log(zw) = \log(1) = \ln(1) + 0i = 0 \]

(b) \( e^{\log(z)} = z, \forall z \in \mathbb{C}, z \neq 0. \)

**Proof. True.**

Write \( z = re^{i\theta}, \) for \( \theta \in (-\pi, \pi] \) and \( r \neq 0. \)

Then
\[ \log(z) = \ln(r) + i\theta \]

so that
\[ e^{\log(z)} = e^{\ln(r) + i\theta} = re^{i\theta} = z. \]

(c) \( \sqrt{z^2 - 1} = \sqrt{z - 1}\sqrt{z + 1}, \forall z \in \mathbb{C}, z \neq \pm1. \)

**Proof. False.**

Let \( z = -2. \) Then
\[ \sqrt{z^2 - 1} = \sqrt{3} \]

But
\[ \sqrt{z - 1} = \sqrt{-3} = i\sqrt{3} \]

and
\[ \sqrt{z + 1} = \sqrt{-1} = i \]

so
\[ \sqrt{z - 1}\sqrt{z + 1} = i^2\sqrt{3} = -\sqrt{3} \]

Therefore,
\[ \sqrt{z^2 - 1} \neq \sqrt{z - 1}\sqrt{z + 1} \]

**Exercise 4.** Determine all \( z \in \mathbb{C} \) such that \( z^2 + 1 \in \mathbb{R}_{\leq 0}. \) Here \( \mathbb{R}_{\leq 0} \) denotes the negative real line.
Proof. Write $z = x + iy$. Then

$$z^2 + 1 = (x^2 - y^2 + 1) + i(2xy)$$

If $z^2 + 1 \in \mathbb{R}_{\leq 0}$, then

$$x^2 - y^2 + 1 \leq 0$$

$$2xy = 0$$

• $x = 0 \implies -y^2 + 1 \leq 0 \implies y \geq 1$ or $y \leq -1$

• $y = 0 \implies x^2 + 1 \leq 0$ a contradiction!

Therefore,

$$z \in \{z = iy \mid y \in \mathbb{R} \text{ and } |y| \geq 1\}$$

\[\square\]

Note: Geometrically, if we write $z = re^{i\theta}$ in polar form, then $z^2 = r^2e^{i(2\theta)}$ and so the module has squared and the angle has doubled. In light of this, if $z^2 + 1 \in \mathbb{R}_{\leq 0}$, then the angle of $z^2 + 1$ is twice of the angle of $z$ since add 1 to $z^2$ only shits $z^2$ horizontally and is nothing to do with the angle. Therefore,

$$2\theta = (2k + 1)\pi, \quad k \in \mathbb{Z}$$

or

$$\theta = (k + \frac{1}{2})\pi, \quad k \in \mathbb{Z}$$

Now, $Re(z^2 + 1) = r^2\cos((2k + 1)\pi) + 1 \leq 0$ implies $-r^2 + 1 \leq 0$ or

$$r^2 \geq 1$$

So

$$z = ir, \quad r \geq 1.$$ 

Exercise 5. Before doing the following problem, please take a look at the supplemental material called ”Mapping properties of the exponential function (continued)” posted on Canvas and course website. Make sure to include the Mathematica codes you use and some brief comments.

(a) Draw the image of the rectangle $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \times [-1, 1]$ under $f(z) = \sin z$. 

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Answer:

\[ f[z_] := \sin[z] \]

\[ \text{ParametricPlot[ReIm[f[t + s*I]], \{t, -Pi + 0.5, Pi + 0.5\}, \{s, -1, 1\}] } \]

(b) Draw the images of some horizontal and vertical lines under \( f \). Based on the picture, can you tell what the images of the line \( x = -\frac{\pi}{2} \) and \( x = \frac{\pi}{2} \) are?

Proof.

\[ \text{ParametricPlot[ReIm[f[t + s*I]], \{t, -Pi + 0.5, Pi + 0.5\}, \{s, -1, 1\}, PlotRange \to \text{Full}, Mesh \to \text{Automatic}, MeshStyle \to \{\text{Red}, \text{Green}\}] } \]

The function \( f \) maps the line \( x = -\frac{\pi}{2} \) to the horizontal half-line: \( x \leq -1 \), and maps the line \( x = \frac{\pi}{2} \) to the half-line: \( x \geq 1 \).

(c) \( f \) maps the vertical strip \( \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right] \times (-\infty, \infty) \) onto some region \( \Omega \). What is \( \Omega \)? Is the map \( f : \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right] \times (-\infty, \infty) \longrightarrow \Omega \) a one-to-one mapping?
Proof. $f$ maps the vertical strip $\left[ -\frac{\pi}{2}, \frac{\pi}{2} \right] \times (-\infty, \infty)$ one-to-one and onto the whole complex plane $\mathbb{C}$.

(d) $f$ maps the vertical strip $\left( -\frac{\pi}{2}, \frac{\pi}{2} \right) \times (-\infty, \infty)$ onto some region $D$. What is $D$? Is the map $f : \left( -\frac{\pi}{2}, \frac{\pi}{2} \right) \times (-\infty, \infty) \rightarrow D$ a one-to-one mapping?

Proof. $f$ maps the vertical strip $\left( -\frac{\pi}{2}, \frac{\pi}{2} \right) \times (-\infty, \infty)$ one-to-one and onto the region: $\mathbb{C} \setminus (\mathbb{R}_{\leq -1} \cup \mathbb{R}_{\geq 1})$
```
In[32]:= p[s_] := ParametricPlot[ReIm[x + y*I], {x, -s, s}, {y, -5, 5},
PlotRange -> {{-2, 2}, {-8, 8}}, MeshStyle -> {Red, Green}, Mesh -> Automatic]

In[33]:= q[s_] := ParametricPlot[ReIm[f[x + y*I]], {x, -s, s}, {y, -5, 5},
PlotRange -> {{-10, 10}, {-10, 10}}, MeshStyle -> {Red, Green},
Mesh -> Automatic]

In[34]:= Manipulate[{p[s], q[s]}, {s, Pi*0.5 - 1, Pi*0.5 + 1}]
```

Out[34]=

```
In[32]:= p[s_] := ParametricPlot[ReIm[x + y*I], {x, -s, s}, {y, -5, 5},
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Out[34]=

In[32]:= p[s_] := ParametricPlot[{Re[x + y I], x, -s, s}, {y, -5, 5},
PlotRange -> {{-2, 2}, {-8, 8}}, MeshStyle -> {Red, Green}, Mesh -> Automatic]

In[33]:= q[s_] := ParametricPlot[{Re[f[x + y I]], x, -s, s}, {y, -5, 5},
PlotRange -> {{-10, 10}, {-10, 10}}, MeshStyle -> {Red, Green},
Mesh -> Automatic]

In[34]:= Manipulate[{p[s], q[s]}, {s, Pi/0.5 - 1, Pi/0.5 + 1}]