Exercise 1. Let $G$ be an open subset of $\mathbb{C}$. Let $f : G \to \mathbb{C}$ be a holomorphic function. Let $a \in G$. Consider the function

$$g(z) = \begin{cases} \frac{f(z) - f(a)}{z - a} & \text{if } z \neq a, \\ f'(a) & \text{if } z = a. \end{cases}$$

(a) Show that $g$ is a continuous function on $G$.

**Answer:** For $z \in G$, $z \neq a$, the function $g(z) = \frac{f(z) - f(a)}{z - a}$ is a multiplication of two continuous functions, namely, $f(z) - f(a)$ and $\frac{1}{z - a}$. Thus $g$ is also continuous at any $z \neq a$. Let us show the continuity of $g$ at $z = a$. We have

$$\lim_{z \to a} g(z) = \lim_{z \to a} \frac{f(z) - f(a)}{z - a} = f'(a) = g(a).$$

Therefore, $g$ is continuous at $z = a$. In conclusion, $g$ is continuous on $G$.

(b) Let us write $f$ in standard form as $f(z) = u(x, y) + iv(x, y)$ where $z = x + iy$. Write $g$ in standard form.

**Answer:** We may assume $a = 0$ for simplicity.

Let us simply write $u(x, y)$ and $u(0, 0)$ as $u$ and $u_0$ respectively. Similarly for $v$ and $v_0$. For $z \neq 0$, we have

$$g(z) = g(z) = \frac{f(z) - f(0)}{z} = \frac{(u + iv) - (u_0 + iv_0)}{x + iy} = \frac{(u - u_0) + iv - v_0}{x + iy}$$

$$= \frac{x(u - u_0) + y(v - v_0)}{x^2 + y^2} + i \frac{x(v - v_0) - y(u - u_0)}{x^2 + y^2}$$

For $z = 0$,

$$g(0) = f'(0) = (\partial_x u)(0, 0) + i (\partial_y v)(0, 0)$$

Therefore,
\[ g(z) = \begin{cases} 
\frac{x(u - u_0) + y(v - v_0)}{x^2 + y^2} + i \frac{x(v - v_0) - y(u - u_0)}{x^2 + y^2} & z \neq 0 \\
(\partial_x u)(0,0) + i(\partial_x v)(0,0) & z = 0.
\end{cases} \]

(c) Show that \( g \) is holomorphic on \( G \).

**Answer:** Because \( f(z) - f(a) \) is differentiable on \( G \) and \( \frac{1}{z-a} \) is differentiable on \( G \setminus \{0\} \), the product \( \frac{f(z) - f(a)}{z-a} \) is differentiable on \( G \setminus \{0\} \). Therefore, \( g \) is differentiable at every \( z_0 \in G \), \( z_0 \neq 0 \).

We now show that \( g \) is differentiable at \( a \). Let us assume \( f(a) = a = 0 \) for simplicity. We will show that the Cauchy–Riemann equations are satisfied at \((0,0)\). Based on the formula we obtained in part (b), \( g(z) = U(x,y) + iV(x,y) \) where

\[
U(x,y) = \begin{cases} 
\frac{xu(x,y) + yv(x,y)}{x^2 + y^2}, & (x,y) \neq (0,0) \\
(\partial_x u)(0,0), & (x,y) = (0,0).
\end{cases}
\]

\[
V(x,y) = \begin{cases} 
\frac{xv(x,y) - yu(x,y)}{x^2 + y^2}, & (x,y) \neq (0,0) \\
(\partial_x v)(0,0), & (x,y) = (0,0).
\end{cases}
\]

Then

\[
\partial_x U(0,0) = \lim_{h \to 0} \frac{U(h,0) - U(0,0)}{h} = \lim_{h \to 0} \frac{\frac{u(h,0)}{h} - \partial_x u(0,0)}{h} = \lim_{h \to 0} \frac{u(h,0) - h\partial_x u(0,0)}{h^2} = \lim_{h \to 0} \frac{\partial_x u(h,0) - \partial_x u(0,0)}{2h} \quad (L'Hopital Rule)
\]

\[
\partial_y U(0,0) = \lim_{h \to 0} \frac{V(0,h) - V(0,0)}{h} = \lim_{h \to 0} \frac{\frac{-u(0,h)}{h} - \partial_y v(0,0)}{h} = \lim_{h \to 0} \frac{-u(0,h) + h\partial_y u(0,0)}{h^2} = \lim_{h \to 0} \frac{\partial_y u(0,h) + \partial_y u(0,0)}{2h} \quad (L'Hopital Rule)
\]
According to Problem 4, \( \partial_{xx} u = -\partial_{yy} u \). Thus, \( \partial_x U(0,0) = \partial_y V(0,0) \). Similarly,

\[
\partial_y U(0,0) = \lim_{h \to 0} \frac{U(0,h) - U(0,0)}{h} = \lim_{h \to 0} \frac{\frac{v(0,h)}{h} - \partial_x u(0,0)}{h} = \lim_{h \to 0} \frac{v(0,h) - h \partial_y v(0,0)}{h^2}, \quad \text{(Cauchy-Riemann eqn’s for } u \text{ and } v) \\
= \lim_{h \to 0} \frac{\partial_y v(0,h) - \partial_y v(0,0)}{2h} = \frac{1}{2} \partial_{yy} v(0,0)
\]

and

\[
\partial_x V(0,0) = \lim_{h \to 0} \frac{V(h,0) - V(0,0)}{h} = \lim_{h \to 0} \frac{\frac{v(h,0)}{h} - \partial_x v(0,0)}{h} = \lim_{h \to 0} \frac{v(h,0) - h \partial_x v(0,0)}{h^2} = \lim_{h \to 0} \frac{\partial_x v(h,0) - \partial_x v(0,0)}{2h} = \frac{1}{2} \partial_{xx} v(0,0) 
\]

According to Problem 4, \( \partial_{xx} v = -\partial_{yy} v \). Thus, \( \partial_y U(0,0) = -\partial_x V(0,0) \).

So \( g \) is differentiable at 0. We have showed that \( g \) is differentiable everywhere in \( G \). We now explain why \( g \) is holomorphic on \( G \).

For each \( z_0 \in G \), there is an open disk \( D_r(z_0) \) centered at \( z_0 \) with radius \( r \) that lies entirely in \( G \). This is because \( G \) is an open set. We know that \( g \) is differentiable everywhere in this disk. Thus, \( g \) is holomorphic at \( z_0 \). Because \( z_0 \) is arbitrary in \( G \), we conclude that \( g \) is holomorphic on \( G \).

\[\square\]

**Exercise 2.** Let \( G \) be an open connected subset of \( \mathbb{C} \). Let \( f : G \to \mathbb{C} \) be a holomorphic map on \( G \). Suppose \( f'(z) = 0 \) for all \( z \in G \). We want to show that \( f \) is a constant function. Follow the steps:

(a) Write \( f \) in standard form \( f(z) = u(x,y) + iv(x,y) \). Show that \( u_x = u_y = v_x = v_y = 0 \) in \( G \).

**Answer:** We know that \( f'(z) = u_x + iv_x \). Therefore, \( u_x = v_x = 0 \) everywhere in \( G \). By Cauchy–Riemann equations, \( u_y = -v_y = 0 \) and \( v_y = u_x = 0 \).

(b) Show that \( u \) and \( v \) are constant functions.
**Answer:** Fix \( z_0 = x_0 + iy_0 \in G \). Let \( w = a + ib \) be an arbitrary point in \( G \). Since \( G \) is open and connected, there is a rectilinear curve lies entirely in \( G \) that starts at \( z_0 \) and ends at \( w \). This rectilinear curve consists of horizontal and vertical line segments, let us denote the joint points by \( z_k = x_k + iy_k \), \( k = 0, 1, \ldots, n \), where \( z_n = a + ib \). (See the graph below)

\[
\text{In other words, each line segment } [z_k, z_{k+1}], k = 0, 1, \ldots, n-1, \text{ is either a horizontal or vertical line.}
\]

If the segment that connect \( z_k \) to \( z_{k+1} \) is horizontal then \( u(x_{k+1}, y_{k+1}) = u(x_k, y_k) \) because \( u_x = 0 \). If the segment that connect \( z_k \) to \( z_{k+1} \) is vertical then \( u(x_{k+1}, y_{k+1}) = u(x_k, y_k) \) because \( u_y = 0 \). Hence,
\[
u(x_{k+1}, y_{k+1}) = u(x_k, y_k)
\]

Since this holds for all \( k \in \{0, 1, 2, \ldots, n\} \), we have
\[
u(x_0, y_0) = u(x_1, y_1) = \cdots = u(x_n, y_n) = u(a, b)
\]

Similarly, \( v(x_0, y_0) = v(a, b) \)

Finally, since this holds for arbitrary \( w = a + ib \in G \), we conclude that \( u \) and \( v \) are constant functions on \( G \). \( \square \)

(c) Is \( f \) necessarily a constant function if the condition "\( G \) is a connected subset" is dropped?

**Answer:** Let
\[
A := \{ z = x + iy : x < -1 \}
\]
and
\[
B := \{ z = x + iy : x > 1 \}
\]
Define \( G := A \cup B \). See Figure 2

\[
\text{RegionPlot} \[ x < -1 || x > 1, \{ x, -3, 3 \}, \{ y, -2, 2 \}, \\
\text{AspectRatio} \to \text{Automatic} \]

4
A point in set $A$ cannot be connected to a point in set $B$ by a path that lies in $G$. Therefore, $G$ is not connected.

Consider the function

$$f(z) = \begin{cases} 
0 & \text{if } z \in A \\
1 & \text{if } z \in B 
\end{cases}$$

We see that $f'(z) = 0$. However, $f$ is not constant in $G$ (although it is constant in $A$ and $B$).

**Exercise 3.** Consider $f(z) = \frac{1}{z}$. We know that $F(z) = \log z$ is an antiderivative of $f$. To be more precisely, $F$ is an antiderivative of $f$ in the region $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$. In this problem, we will see that $f$ has many other antiderivatives (differing $f$ by a non-constant) in other regions.

(a) Show that any antiderivative of $f$ in $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$ must be $F(z) + c$ where $c$ is a complex constant.

**Answer:** Let $G(z)$ be any antiderivative of $f$ in $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$. Note that $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$ is open and connected and $G$ is holomorphic on $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$. Then

$$(G - F)'(z) = G'(z) - F'(z) = \frac{1}{z} - \frac{1}{z} = 0$$

Since $G(z) - F(z)$ is also holomorphic on $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$, Problem 2 tells us that $G(z) - F(z)$ is constant, say, $c$. Thus, $G(z) - F(z) = c$ or $G(z) = F(z) + c$.

(b) For $\theta \in (-\pi, \pi]$, denote $G(z) = \log (e^{i\theta}z)$. Describe the region of continuity of $G$. Show that $G' = f$ in this region. Is the difference $G - F$ a constant function?

**Answer:** $\log(e^{i\theta}z)$ is discontinuous at those $z$ such that

$$e^{i\theta}z \in \mathbb{R}_{\leq 0}$$

write $z = re^{i\beta}$, then
\[
e^{i\theta}z = re^{i(\theta+\beta)} \in \mathbb{R}_{\leq 0} \iff \theta + \beta = \pi \pmod{2\pi}
\]

Thus, \(G\) is continuous everywhere except on the ray \(\text{Arg} z = \pi - \theta\) (the green line in Figure 2):

![Figure 2:](image)

Using differentiation rule
\[
G'(z) = (\log(e^{i\theta}z))' = \frac{1}{e^{i\theta}z}e^{i\theta} = \frac{1}{z}
\]

which shows that \(G' = f\) in the region \(\mathbb{C} \setminus \{z = re^{i\beta} : \beta = \pi - \theta\}\).

\(G - F\) is a holomorphic function in the region \(\Omega = A \cup B\) as shown in Figure 2. It is not a constant function on \(\Omega\) unless \(\theta = 0\). To see this, let us take \(\theta = \pi\) for example.

For \(z_1 = e^{-i\pi/2}\), we have
\[
\log(e^{i\theta}z_1) = \log(e^{i\pi/2}) = i\frac{\pi}{2}
\]
so \(G(z_1) - F(z_1) = i\frac{\pi}{2} - i\frac{-\pi}{2} = i\pi\).

For \(z_2 = e^{-i3\pi/2}\), we have
\[
\log(e^{i\theta}z_2) = \log(e^{-i\pi/2}) = -i\frac{\pi}{2}
\]
so \(G(z_2) - F(z_2) = -i\frac{\pi}{2} - i\frac{\pi}{2} = -i\pi\).

Thus, \(G - F\) is not a constant function on \(\Omega\). However, it is a constant function on \(A\) and on \(B\). Note that \(\Omega\) is not a connected set because it is impossible to connect a point in \(A\) to a point in \(B\) by any continuous path. This is another example (in addition to the example given in Problem 2, Part (c)) to show that a function whose derivative is equal to zero on a disconnected set may not be a constant function.

(c) Show that \(f\) has no antiderivatives in the region \(\mathbb{C} \setminus \{0\}\).

**Answer:** Suppose by contradiction that there is an antiderivative of \(f\) in the region \(\mathbb{C} \setminus \{0\}\). Let us call it \(H(z)\). We have \(H'(z) = f(z)\) for all \(z \in \mathbb{C} \setminus \{0\}\).
In particular, $H(z)$ is an antiderivative of $f(z)$ in the region $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$. According to Part (a), $H(z) = F(z) + c$ in the region $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$. Because $F$ is discontinuous on the negative real line (jumping by $2\pi i$ across the negative real line), so must be $H$. On the other hand, $H$ is holomorphic everywhere on $\mathbb{R}_{< 0}$, so it must be continuous everywhere on $\mathbb{R}_{< 0}$. This is a contradiction.

A function $u(x, y)$ is said to be **harmonic** in a region $G$ if the Laplacian $\Delta u = u_{xx} + u_{yy}$ is equal to zero for all $(x, y) \in G$.

**Exercise 4.** Let $f(z) : G \to \mathbb{C}$ be a holomorphic function on $G$. Show that the real part and imaginary part of $f$ are harmonic functions.

**Answer:** Write $f(z) = u(x, y) + iv(x, y)$.
Since $f$ is differentiable in $G$, the Cauchy-Riemann equations holds:

$$u_x = v_y; \quad u_y = -v_x$$

Thus,

$$u_{xx} + u_{yy} = v_{xy} + (-v_{yx}) = 0$$

and

$$v_{xx} + v_{yy} = -u_{xy} + u_{yx} = 0$$

**Exercise 5.** Find an entire function $f$ such that $f(0) = 1 - 2i$ and the real part of $f$ is $u(x, y) = e^{-y} \cos(x) - y$.

**Answer:** Let $v(x, y)$ be the imaginary part of $f(z)$. Since $f$ is entire, it must satisfy the Cauchy-Riemann equations. Thus,

$$v_x = -u_y = e^{-y} \cos(x) + 1$$

and

$$v_y = u_x = -e^{-y} \sin(x)$$

Integrating (5) with respect to $x$, we get

$$v(x, y) = e^{-y} \sin(x) + x + C(y).$$

Differentiate both sides with respect to $y$:

$$v_y = -e^{-y} \sin(x) + C'(y).$$

Comparing this equation with (6), we get $C'(y) = 0$. Thus, $C(y) = c$ for some constant $c$. We obtain

$$v(x, y) = e^{-y} \sin(x) + x + c$$
To find \( c \), we use the fact that \( f(0) = 1 - 2i \). This equation implies \( c = -2 \).
Therefore,

\[
v(x, y) = e^{-y} \sin(x) + x - 2
\]

So,

\[
f(z) = (e^{-y} \cos(x) - y) + i (e^{-y} \sin(x) + x - 2)
\]

If one wishes to obtain a neat formula in terms of \( z \), one can proceed as follows:

\[
f(z) = e^{-y}(\cos x + i \sin x) + (-y + ix) - 2i \\
= e^{-y+ix} + (-y + ix) - 2i \\
= e^{i(x+iy)} + i(x + iy) - 2i \\
= e^{ix} + iz - 2i.
\]

The above procedure is simply cosmetic!