A subset $G \subset \mathbb{C}$ is said to be open if for every $z \in G$ there is a disk $D_r(z)$ (centered at $z$ with radius $r > 0$) that lies entirely in $G$.

1. Let $G$ be an open subset of $\mathbb{C}$. Let $f : G \to \mathbb{C}$ be a holomorphic function. Let $a \in G$. Consider the function

$$g(z) = \begin{cases} \frac{f(z) - f(a)}{z - a} & \text{if } z \neq a, \\ f'(a) & \text{if } z = a. \end{cases}$$

(a) Show that $g$ is a continuous function on $G$.

(b) Let us write $f$ in standard form as $f(z) = u(x,y) + iv(x,y)$ where $z = x + iy$. Write $g$ in standard form. (For the sake of simplicity, you can assume $f(a) = a = 0$.)

(c) Show that $g$ is holomorphic on $G$.

Hint: explain why $g$ is differentiable on $D \setminus \{a\}$. Then use Cauchy–Riemann equations to explain why $g$ is differentiable at $a$. (For the sake of simplicity, you can assume $f(a) = a = 0$.)

An open subset $G \subset \mathbb{C}$ is said to be connected if for any two points $z_1, z_2 \in G$ there is a rectilinear curve inside $G$ that starts at $z_1$ and ends at $z_2$. A rectilinear curve is a curve that consists of horizontal and vertical line segments. See the figure.

2. Let $G$ be an open and connected subset of $\mathbb{C}$. Let $f : G \to \mathbb{C}$ be a holomorphic map on $G$. Suppose $f'(z) = 0$ for all $z \in G$. We want to show that $f$ is a constant function. Follow the steps:

(a) Write $f$ in standard form $f(z) = u(x,y) + iv(x,y)$. Show that $u_x = u_y = v_x = v_y = 0$ in $G$.

(b) Show that $u$ and $v$ are constant functions.

Hint: Fix a point $z_0 = x_0 + iy_0$ in $G$. Let $w = a + ib$ be an arbitrary point in $G$. Explain why $u(a,b) = u(x_0,y_0)$. Explain likewise for $v$.

(c) Is $f$ necessarily a constant function if the condition “$G$ is a connected subset” is removed?

3. Consider $f(z) = \frac{1}{z}$. We know that $F(z) = \log z$ is an antiderivative of $f$. To be more precisely, $F$ is an antiderivative of $f$ in the region $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$. In this problem, we will see that $f$ has many other antiderivatives (differing $f$ by a non-constant) in other regions.

(a) Show that any antiderivative of $f$ in $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$ must be $F(z) + c$ where $c$ is a complex constant.

Hint: use Problem 2.
(b) For $\theta \in (-\pi, \pi]$, denote $G(z) = \log(e^{i\theta}z)$. Describe the region of continuity of $G$. Show that $G' = f$ in this region. Is the difference $G - F$ a constant function?

(c) Show that $f$ has no antiderivatives in the region $\mathbb{C}\{0\}$.

Hint: suppose by contradiction that it has an antiderivative $H$ in this region. Use Part (a) to show that $R(z) = H(z) - F(z)$ is constant on $\mathbb{C}\{0\}$.

A function $u(x,y)$ is said to be harmonic in a region $G$ if the Laplacian $\Delta u = u_{xx} + u_{yy}$ is equal to zero for all $(x,y) \in G$.

4. Let $f : G \to \mathbb{C}$ be a holomorphic function on $G$. Show that the real part and imaginary part of $f$ are harmonic functions.

5. Find an entire function $f$ (i.e. $f$ is holomorphic on $\mathbb{C}$) such that $f(0) = 1 - 2i$ and the real part of $f$ is

$$u(x,y) = e^{-y}\cos(x) - y.$$