Lecture 13
Monday, April 27, 2020

\( f \) is cont. at \( z_0 \) if

\[
\lim_{z \to z_0} f(z) = f(z_0)
\]

\[ f(z) = u(z) + iv(z) \]

Ref \( f \) Im \( f \)

\[
\lim_{z \to z_0} f(z) = f(z_0)
\]

\[
\lim_{z_0 \to z_0} u(z_0) = u(z_0)
\]

\[
\lim_{z_0 \to z_0} v(z_0) = v(z_0)
\]

\( u, v \) are cont. at \( z_0 \).

Ex: \( f(z) = z^2 \)

\( z = x + iy \)

\[
f(z) = (x + iy)^2 = x^2 - y^2 + i2xy
\]

\( u \) can be considered as \( u(x, y) = x^2 - y^2 \)

\( v \) is cont. if \( v(x, y) = 2xy \).

Conclude: \( f(z) = z^2 \) is cont. at every \( z \).
\[ f(z) = \frac{1}{z} = u(z) + i v(z) \]

\[
\begin{align*}
\{ u(z) &= |z| \\
v(z) &= 0
\end{align*}
\]

\[ u(x, y) = \sqrt{x^2 + y^2} \]

Continous. (Mark 2.5)

\[ f(z) = \frac{b}{|z|^2} = u(z) + i v(z). \]

\[ z = x + iy \]

\[ \overline{z} = x - iy \]

\[ f(z) = \frac{x + iy}{\sqrt{x^2 + y^2}} = \frac{x}{\sqrt{x^2 + y^2}} + i \left( \frac{-y}{\sqrt{x^2 + y^2}} \right) \]

\[ u(x, y) \] and \[ v(x, y) \] are cont at any point \((x, y) \neq (0, 0)\)

\[ u(x, y) = \frac{x}{\sqrt{x^2 + y^2}} \]

\[ (x_0, y_0) = \left( \frac{1}{n}, 0 \right) \quad \Rightarrow \quad u(x_0, y_0) = \frac{1/n}{\sqrt{1/n^2}} = \frac{1}{n} \]

\[ (x_0, y_0) = \left( -\frac{1}{n}, 0 \right) \quad \Rightarrow \quad u(x_0, y_0) = -\frac{1/n}{1/n} = -1 \]

\[ u \text{ is Cont. at } (0,0). \]

\[ \text{\underline{\text{\textit{f is Dis. at } z = 0.}}} \]

\[ f(z) = 0 \]

\[ z = x + iy \]

\[ f(z) = x + iy \]

\[ \{ u(x, y) = x \quad \rightarrow \text{Cont.} \]

(Marked 2.5)
Fix \( x_0 \in \mathbb{C} \).
Let \((x_n)\) be a sequence that converges to \( x_0 \),
\[
f(x_n) = z_n \to z_0 = f(x_0)
\]
\( f \) is cont. at \( x_0 \).

Rule: If \( f \) and \( g \) are cont. at \( x_0 \), then \( f \circ g \) is cont. at \( x_0 \).

If \( f \) is cont. at \( y_0 \) and \( g(x_0) = y_0 \), then \( f \circ g \) is cont. at \( x_0 \).

\( f(x) = z \quad \text{-- continuous} \)
\( f(x) = c \quad (\text{constant}) \quad \text{-- continuous} \)
\( \frac{z^2 + z + 1}{z^2 + 2z + 1} \quad \text{-- continuous (at any point)} \)

Any poly. is a cont. function.
\(x < 0.\)

We want to show that \(f(z) = \text{Arg}(z)\)

is discontinuous at \((x,0)\).

Consider \(z_n = x + i \frac{1}{n}\).

\[
\begin{align*}
\lim_{n \to \infty} f(z_n) &= \text{Arg}(x + i \frac{1}{n}) \to \pi. \\
\end{align*}
\]

\[
\begin{align*}
\lim_{n \to \infty} f(z_n) &= \text{Arg}(x - i \frac{1}{n}) \to -\pi.
\end{align*}
\]
For your thoughts:

$f(z) = \log z \ldots \text{where is } f \text{ unit?}$

$f(z) = \sqrt{z^2 - 1} \ldots " \quad "$

principal logarithm:

$\sqrt{z} = e^{\frac{1}{2} \log z}$