A complex number \( z \) can be represented in standard form \( z = a + ib \) or polar form \( z = re^{i\theta} \). One form can be converted to the other. Given the polar form, it is quite easy to get the standard form because

\[
\begin{align*}
a &= r \cos \theta, \\
b &= r \sin \theta.
\end{align*}
\]

Given the standard form, one can find \( r = \sqrt{a^2 + b^2} \). The argument \( \theta \) is a little more tricky:

\[
\begin{align*}
\cos \theta &= \frac{a}{r} = \frac{a}{\sqrt{a^2 + b^2}}, \\
\sin \theta &= \frac{b}{r} = \frac{b}{\sqrt{a^2 + b^2}}.
\end{align*}
\]

Let’s consider an example \( z = -1 + 2i \).

\[
r = |z| = \sqrt{(-1)^2 + 2^2} = \sqrt{5}.
\]

\[
\cos \theta = -\frac{1}{\sqrt{5}}, \quad \sin \theta = \frac{2}{\sqrt{5}}.
\]

We see that the angle \( \theta \) lies in between \( \frac{3\pi}{2} \) and \( \pi \). The \( \arccos \) function goes values in the range \([0, \pi]\). Thus,

\[
\theta = \arccos \left( -\frac{1}{\sqrt{5}} \right) = 2.0344439....
\]

Let’s consider another example \( w = -1 - 2i \).

\[
r = \sqrt{(-1)^2 + (-2)^2} = \sqrt{5}
\]

\[
\cos \theta = -\frac{1}{\sqrt{5}}, \quad \sin \theta = -\frac{2}{\sqrt{5}}.
\]

\( \theta \) is in the range of \([\pi, \frac{3\pi}{2}]\) according to the picture. Thus,

\[
\theta = -\arccos \left( \frac{1}{\sqrt{5}} \right) = -2.034444....
If \( z = r e^{i\theta} \) then \( \theta \) is called an argument of \( z \). It is unique modulo \( 2\pi \).

\[
\arg(z) = \theta + 2k\pi \quad (k \in \mathbb{Z})
\]

\[
= \{ \theta, \theta + 2\pi, \theta - 2\pi, \theta + 4\pi, \ldots \}
\]

"arg" is a multi-valued “function”. It is not a function in usual sense. It has connection with the logarithm function. We will discuss in detail later.

\( \text{Arg}(z) \) is the value of argument that lies in \( (-\pi, \pi] \). It is called the principal argument of \( z \).

\( \text{Ex:} \)

\[
\arg(-1+2i) = 2.0344... + k2\pi \quad (k \in \mathbb{Z})
\]

\[
\text{Arg}(-1+2i) = 2.0344... \quad \in [-\pi, \pi)
\]

\[
\arg(-1-2i) = -2.0344... + k2\pi \quad (k \in \mathbb{Z})
\]

\[
\text{Arg}(-1-2i) = -2.0344...
\]

* Complex conjugate:

For \( z = a+ib \), the number \( \bar{z} = a-ib \) is called is called the complex conjugate of \( z \).

\[
\text{Geometrically,} \ z \ \text{and} \ \bar{z} \ \text{are mirror reflection of each other with respect to the x-axis.}
\]

\[
|z| = |\bar{z}| = r = \sqrt{a^2 + b^2}
\]

If \( \theta \) is an argument of \( z \) then \(-\theta\) is an argument of \( \bar{z} \). We also see that

\[
\bar{z} = (a+bi)(a-ib) = a^2 + b^2 = |z|^2.
\]

* Geometric interpretation of complex multiplication:

Consider two complex numbers written in standard form

\[
z = a+ib, \quad w = c+id.
\]
We know how to multiply them using the rule $i^2 = -1$.

$$zw = ac - bd + i(ad + bc)$$

The geometric meaning of multiplication is more clear in the polar form. Write $z = r e^{i\theta}$ and $w = s e^{i\phi}$. Then

$$zw = (re^{i\theta})(se^{i\phi}) = rs e^{i(\theta + \phi)}.$$

We have

$$e^{i\theta} e^{i\phi} = (\cos \theta + i\sin \theta)(\cos \phi + i\sin \phi)$$

$$= (\cos \theta \cos \phi - \sin \theta \sin \phi) + i(\sin \theta \cos \phi + \cos \theta \sin \phi)$$

$$= \cos(\theta + \phi) + i\sin(\theta + \phi)$$

$$= e^{i(\theta + \phi)}.$$

Therefore,

$$zw = rs e^{i(\theta + \phi)}.$$  \(\star\)

The rule of thumb is: to multiply two complex numbers, we multiply their modules to get the modulus, and add their arguments to get the argument.

Formula \(\star\) has a helpful consequence. Suppose we are to compute \((-1+2i)^{10}\).

Of course we can multiply $-1+2i$ by itself ten times, using distribution rule and the rule $i^2 = -1$. This practice takes a long time. A more efficient way is as follows.

First, we express $z = -1 + 2i$ in polar form: $z = \sqrt{5} e^{i\theta}$.

Then we raise $z$ to power 10 by raising $r = \sqrt{5}$ by power 10 and multiplying $\theta = 2.0344\ldots$ by 10.

$$z^{10} = (\sqrt{5})^{10} e^{i20.344\ldots} = 3125 e^{i20.344\ldots}$$

polar form of $z^{10}$
In general, if \( n \) is an integer then
\[
(r e^{i\theta})^n = r^n e^{i\theta n}
\]
This is known as de Moivre's formula (~1722).

Now how do we take roots of a complex number? For example, how do we find \( \sqrt[3]{-1+2i} \)?

Let us put \( w = \sqrt[3]{-1+2i} \). We want to solve for complex number \( w \) from the equation \( w^3 = -1+2i \). Write \( w \) in polar form:
\[
w = r e^{i\theta}
\]
where \( r \) and \( \theta \) are to be determined. We also write \(-1+2i\) in polar form:
\[
-1+2i = \sqrt{5} e^{i\theta}
\]
where \( r = \sqrt{5} \) and \( \theta = 2.0344... \) (as computed earlier). The equation \( w^3 = -1+2i \) becomes
\[
s^3 e^{i3\theta} = \sqrt{5} e^{i\theta}.
\]
For two complex numbers to be equal to each other, the moduli must be equal and the arguments must be equal in modulo \( 2\pi \).
Thus,
\[
\begin{align*}
s^3 &= \sqrt{5} \\
3\theta &= \theta + 2k\pi \quad \text{(for some integer } k)\end{align*}
\]
We get
\[
\begin{align*}
s &= \sqrt[3]{5} \quad \text{(regular third root of a real number)} \\
\theta &= \frac{\theta}{3} + k \frac{2\pi}{3}.
\end{align*}
\]
We see that there are multiple values of \( w \) because there are multiple values of \( \theta \). Let us put the values of \( w \) on the complex plane.

We see that there are only 3 values of \( \theta \). They are equally spaced on the circle of radius \( s = \sqrt{5} \).
In conclusion,
\[ \sqrt[4]{-1 + 2i} = \sqrt[5]{e^{i(\frac{\pi}{4} + k \frac{2\pi}{5})}} \] with \( k = 0, 1, 2 \).

One can use the same method to find the \( n \)-th root of a complex number \( z = r e^{i \theta} \).
\[ \sqrt[n]{z} = \sqrt[n]{r} e^{i \left( \frac{\theta}{n} + k \frac{2\pi}{n} \right)} \] with \( k = 0, 1, 2, \ldots, n-1 \).