As we saw last time, any nonzero complex number has exactly n
n-th roots. Zero has only one n-th root, which is zero. In polar form,
z = r e^{i\theta} and
\[ z^{\text{1/n}} = \sqrt[n]{r} \ e^{i\left(\frac{\theta + 2k\pi}{n}\right)} \]
for \( k = 0, 1, 2, \ldots, n-1 \).

Root of a real number

Geometrically, these n-th roots of z lie on a circle of radius \( \sqrt[n]{r} \) and
are equally spaced, starting with angle \( \frac{\theta}{n} \), forming a regular n-polygon.

Ex: Find all third roots of 1 and locate them on the complex plane.

In polar form, \( 1 = 1 e^{i0} \). Thus,
\[ 1^{1/3} = \sqrt[3]{1} e^{i\left(\frac{0 + 2k\pi}{3}\right)} = e^{ik\frac{2\pi}{3}} \]
for \( k = 0, 1, 2 \).

We have seen that the arithmetic of complex numbers is very much the
same as the arithmetic of real numbers. The number i was introduced
somewhat artificially: a “number” satisfying \( i^2 = -1 \). In the following
is an example of a structure that is essentially the same as complex numbers,
where the imaginary unit is a natural (as oppose to artificial) element.
Consider the set of matrices

\[ M = \left\{ \begin{bmatrix} a & -b \\ b & a \end{bmatrix} : a, b \in \mathbb{R} \right\} . \]

M doesn't contain all 2x2 matrices, only those of the form \( \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \).

M has vector space structure:

- The sum of two elements of M is another element of M:
  \[ \begin{bmatrix} a & -b \\ b & a \end{bmatrix} + \begin{bmatrix} c & -d \\ d & c \end{bmatrix} = \begin{bmatrix} a+c & -(b+d) \\ b+d & a+c \end{bmatrix} \in M. \]

- The (real) scaling of an element of M is another element of M:
  \[ \alpha \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} \alpha a & -\alpha b \\ \alpha b & \alpha a \end{bmatrix} \in M. \]

M is also closed under multiplication:

\[ \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} c & -d \\ d & c \end{bmatrix} = \begin{bmatrix} ac-bd & -(ad+bc) \\ ad+bc & ac-bd \end{bmatrix} \in M. \]

M is also closed under division (inversion):

\[ \begin{bmatrix} a & -b \\ b & a \end{bmatrix}^{-1} = \frac{1}{a^2+b^2} \begin{bmatrix} a & b \\ -b & a \end{bmatrix} = \begin{bmatrix} a/(a^2+b^2) & b/(a^2+b^2) \\ -b/(a^2+b^2) & a/(a^2+b^2) \end{bmatrix} \in M. \]

We see that matrix \( \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \) behaves just like complex number \( a+ib \). [One can check how similar M is to \( \mathbb{C} \) by looking at the multiplication rule and inversion rule above.]

Matrix \( \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \) behaves like number 1 \( \in \mathbb{C} \) (the multiplication unit). Matrix \( \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \) behaves like number \( i \in \mathbb{C} \) because

\[ \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \]

Also,
\[
\begin{bmatrix}
a & -b \\
b & a
\end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}
\] is like “\(a + bi\).”

Those who already took Math 343 can see that \(M\) is a field and is isomorphic to \(\mathbb{C}\), i.e. having the same algebraic structure. In other words, \(M\) is another manifestation of complex numbers. The “imaginary” number \(i\) has become so “real.”

* Solving quadratic equations:

Recall how we solved the quadratic equation

\[ax^2 + bx + c = 0\]

where \(a, b, c \in \mathbb{R}\). The main idea is to complete the square:

\[
\begin{align*}
\frac{b}{a}x + c &= 0 \\
\Rightarrow &
\end{align*}
\]

\[
(x + \frac{b}{2a})^2 + \frac{c}{a} = \frac{b^2}{4a^2}
\]

Then we get

\[
(x + \frac{b}{2a})^2 = \frac{b^2 - 4ac}{4a^2} = \frac{\Delta}{4a^2} \tag{*}
\]

This equation has two distinct real roots if \(\Delta > 0\), a double root if \(\Delta = 0\), and no real roots if \(\Delta < 0\).

If we allow \(x\) to be complex, we don’t need to worry about the sign of the discriminant \(\Delta\).

\[
x = \frac{-b \pm \sqrt{\Delta}}{2a}
\]

or

\[
x = \frac{-b \pm \sqrt{\Delta}}{2a} \quad \text{(quadratic formula)}
\]

Here \(a, b, c\) can be complex numbers. Note that \(\Delta\) has two complex square roots differing each other by a sign, one can write the above...
formula as \[ x = \frac{-b + \sqrt{\Delta}}{2a} \]

with the understanding that \( \sqrt{\Delta} \) has two complex values.

Ex.:

Solve for all complex roots of \( z^5 + z^4 + z^3 + z^2 + z + 1 = 0 \).

We have \( \Delta = 1^5 - 4 \times 1 = -15 \).

\[ \sqrt{\Delta} = \pm i \sqrt{15} \cdot \]

Then \[ z = \frac{-b + \sqrt{\Delta}}{2a} = \frac{-1 \pm i \sqrt{15}}{2} = -\frac{1}{2} \pm i \frac{\sqrt{15}}{2} \]

Ex.:

Solve for all complex roots of \( z^5 - 2z^4 + z^2 - 2 - i = 0 \), with a hint that \( z = i \) is one of the roots.

Because \( i \) is a root, \( z - i \) is a factor of the given cubic polynomial. We will do long division:

\[
\begin{array}{c|ccccc}
& z & -i & \hline
\end{array}
\]

\[ z^5 - 2z^4 + z^3 + z^2 + z + 1 = 0 \]

\[ \frac{z^4 - 2z^3 + 2z - 2 - i}{(z - i)} \]

\[ \frac{z^3 - z^2 - 2z + 2}{(z - i)(z^4 - 2z^3 + 2z - 2 - i)} \]

\[ \frac{z^2 + (2 + i)z + (i + 1)}{(z - i)(z^4 - 2z^3 + 2z - 2 - i)} \]

\[ \frac{z + 2 - i}{(z - i)(z^4 - 2z^3 + 2z - 2 - i)} \]

Thus, \( z^5 - 2z^4 + z^3 + z^2 + z + 1 = (z - i)(z^4 + (-2 + i)z^3 + (1 - 2i)) \).

Of course \( z = i \) is a root. To find other roots, we only need to solve the quadratic equation

\[ z^2 + (-2 + i)z + 1 - 2i = 0 \]

We have

\[ \Delta = (-2 + i)^2 - 4(1 - 2i) = -1 + 4i \cdot \]

To find the square roots of \( \Delta \), we write \( \Delta \) in polar form:
\[ \Delta = -1 + 4i = \sqrt[4]{\left( -\frac{1}{\sqrt{17}} + i \frac{4}{\sqrt{17}} \right)} = r e^{i\theta} \]

with \( r = \sqrt[4]{17} \) and \( \theta = \arccos \left( -\frac{1}{\sqrt{17}} \right) \approx 1.8158. \)

Then \( \sqrt[4]{\Delta} = \pm \sqrt[4]{17} e^{i\theta/4} = \pm \sqrt[4]{17} (\cos \frac{\theta}{4} + i \sin \frac{\theta}{4}) \approx \pm (1.2496 + i 1.6065) \)

One can easily continue from here.

Ex: Solve for all complex roots of the equation

\[ z^4 + z^2 + 1 = 0. \]

But \( w = z^2. \) Then we get \( w^2 + w + 1 = 0. \)

This equation has two roots: \( w_1 = \frac{-1 + i\sqrt{3}}{2} \) and \( w_2 = \frac{-1 - i\sqrt{3}}{2}. \)

Taking the square roots of \( w_1 \) and \( w_2, \) we get \( \xi. \) There are four values of \( \xi, \) two coming from \( w_1 \) and two coming from \( w_2. \)

\[ \sqrt[4]{w_1} = \{ \xi_1, \xi_2 \}, \]
\[ \sqrt[4]{w_2} = \{ \xi_3, \xi_4 \}. \]

We will show later in the course that a polynomial of degree \( n \) always have \( n \) complex roots. This is known as the Fundamental Theorem of Algebra.

* Use Mathematica to solve an equation:

One can use the command `Solve` in Mathematica to find roots of an equation. For example,

\[ \text{Solve}[z^2 + z + 4 == 0, z] \]

(Note that one needs to write two equal signs.)

To solve for numeric values of roots, one can use the command `NSolve`.

For example,

\[ \text{NSolve}[z^3 - 2*z^2 + 2*z - 2 - 1 == 0, z] \]