Last time we discussed how to solve a general quadratic equation of the form $ax^2 + bx + c = 0$. Now we will discuss how to solve a general cubic equation of the form $ax^3 + bx^2 + cx + d = 0$.

If this equation has a special root, say $x = r_0$, then one should factor out $x - r_0$, and then the problem becomes solving a quadratic equation (see an example given last time). Here we are interested in a situation where no special roots are known, for example, the equation

$$x^3 + 2x + 2 = 0.$$ 

One can check that none of the numbers $\pm 1, \pm 2$ solve the equation.

We discuss in Lecture 2 how to solve the cubic equation of the form

$$x^3 + px + q = 0. \quad (*)$$

We refer to that method as Cardano’s method (~1545). His idea is as follows: split $x = uv$ where $u$ and $v$ are to be determined. The equation (*) becomes

$$u^3 + v^3 + (3uv + p)(uv) + q = 0.$$ 

We then imposed an addition on $u$ and $v$, namely $3uv + p = 0$. Then we get a system of two unknowns:

$$\begin{cases} u^3 + v^3 + q = 0 \\ 3uv + p = 0. \end{cases}$$

From here we get

$$\begin{cases} u^3 + v^3 = -q, \\ u^3 v^3 = -p^2/27. \end{cases}$$

The $t = u^3$ and $s = v^3$ solve the quadratic equation

$$w^2 + qw - p^3/27 = 0.$$ 

We get

$$w = \frac{1}{2} \left(-q \pm \sqrt{\Delta}\right) \quad \text{where} \quad \Delta = q^2 + \frac{4p^3}{27}.$$
We get \[ u^3 = \frac{1}{2} (-q + \sqrt{\Delta}) \]
\[ v^3 = \frac{1}{2} (-q - \sqrt{\Delta}) \]

Then \[ u = \sqrt[3]{\frac{1}{2} (-q + \sqrt{\Delta})}, \]
\[ v = \sqrt[3]{\frac{1}{2} (-q - \sqrt{\Delta})}. \]

Note that there are three values of \( u \) and three values of \( v \) because a complex number has three third roots. Let us denote the three values of \( u \) as \( u_1, u_2, u_3 \), and the three values of \( v \) as \( v_1, v_2, v_3 \). The roots \( \tau = u + v \) of the original cubic equation are therefore
\[ u_1 + v_1, \quad u_2 + v_2, \quad u_3 + v_3, \]
\[ u_1 + v_2, \quad u_2 + v_3, \quad u_3 + v_1, \]
\[ u_1 + v_3, \quad u_2 + v_1, \quad u_3 + v_2. \]

Among these 9 values, there are at most 3 distinct values. The remaining 6 are repetitions of these 3 values. One can take the values on the first column or the first row.

If the equation is in a general form \( ax^3 + bx^2 + cx + d = 0 \), we can divide both sides by \( a \):
\[ \frac{x^3}{a} + \frac{b}{a} x^2 + \frac{c}{a} x + \frac{d}{a} = 0 \]

and then use the change of variable \( \tau = z + \frac{b}{3a} \). The equation that \( \tau \) satisfies will be of the form \( \tau^3 + px + q = 0 \).

\[ \text{Ex:} \]
\[ 3z^3 + 3z + 1 = 0 \]

Change of variable \( z = \tau + \frac{1}{3} = \tau + 1 \), or equivalently \( \tau = z - 1 \). Substitute this \( \tau \) into the equation:
\[
(x-1)^3 + 3(x-1)^2 + 1 = 0
\]

or equivalently,
\[
x^3 - 3x + 3 = 0
\]

We know how to solve this equation by Cardano's method.

[See Problem 2 on the worksheet for example.]

The exponential function with complex variable.

We know what \( e^x \) is when \( x \in \mathbb{R} \). That is
\[
e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots
\]

We know what \( e^{iy} \) is when \( y \in \mathbb{R} \). That is
\[
e^{iy} = 1 + \frac{iy}{1!} + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} + \cdots
\]

\[
= \cos y + i \sin y.
\]

How do we define \( e^z \) where \( z = x + iy \)?

There are two equivalent ways to define \( e^z \). The first way is
\[
e^z = e^{x+iy} = e^x e^{iy} = e^x \cos y + i e^x \sin y.
\]

The second way is
\[
e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots
\]

Why are these two ways equivalent to each other? Let us put \( u = x \) and \( v = iy \). We know that
\[
e^u = 1 + \frac{u}{1!} + \frac{u^2}{2!} + \frac{u^3}{3!} + \cdots
\]
\[
e^v = 1 + \frac{v}{1!} + \frac{v^2}{2!} + \frac{v^3}{3!} + \cdots
\]

Then
\[
e^u e^v = \left(1 + \frac{u}{1!} + \frac{u^2}{2!} + \frac{u^3}{3!} + \cdots\right)\left(1 + \frac{v}{1!} + \frac{v^2}{2!} + \frac{v^3}{3!} + \cdots\right)
\]
\[ \begin{align*}
  &= l + (u+v) + \left( \frac{u^2}{2!} + uv + \frac{v^2}{2!} \right) + \cdots \\
  &= 1 + \frac{u+v}{1!} + \sum_{2} \frac{(u+v)^2}{2!} + \cdots
\end{align*} \]

Thus, 
\[ e^x e^y = e^{x+y} = 1 + \frac{u+v}{1!} + \sum_{2} \frac{(u+v)^2}{2!} + \cdots = 1 + \frac{u}{1!} + \frac{v}{2!} + \cdots. \]

How do we define sine and cosine of complex variables? 

We know that \( e^{ix} = \cos x + i\sin x \) for any \( x \in \mathbb{R} \).

Now replace \( x \) by \( -x \):

\[ e^{-ix} = \cos(-x) + i\sin(-x) \]

\[ = \cos x - i\sin x. \]

From here we get

\[ \cos x = \frac{e^x + e^{-x}}{2}, \quad \sin x = \frac{e^x - e^{-x}}{2i}. \]

These identities are true when \( x \) is a real number. We will now allow \( x \) to be any complex number. Thus,

\[ \cos x \overset{\text{def}}{=} \frac{e^x + e^{-i x}}{2}, \quad \sin x = \frac{e^x - e^{-i x}}{2i}. \]

These definitions are the same as

\[ \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots \]

\[ \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \]

(One can check by writing the series for \( e^{ix} \) and \( e^{-ix} \).)

See computation examples on the worksheet.