Last time we discussed how to solve a general quadratic equation of the form \( ax^2 + bx + c = 0 \). Now we will discuss how to solve a general cubic equation of the form \( ax^3 + bx^2 + cx + d = 0 \).

If this equation has a special root, say \( x = x_0 \), then one should factor out \( x - x_0 \), and then the problem becomes solving a quadratic equation (see an example given last time). Here we are interested in a situation where no special roots are known, for example, the equation
\[ x^3 + 2x^2 + 2 = 0. \]
One can check that none of the numbers \( \pm 1, \pm 2 \) solve the equation.

We discuss in Lecture 2 how to solve the cubic equation of the form
\[ x^3 + px + q = 0. \quad (*) \]
We refer to that method as Cardano’s method (≈ 1545). His idea is as follows: split \( x = uv \) where \( u \) and \( v \) are to be determined.
The equation (*) becomes
\[ u^3 + v^3 + (3uv + p)(u + v) + q = 0. \]
We then imposed an addition on \( u \) and \( v \), namely \( 3uv + p = 0 \).
Then we get a system of two unknowns:
\[ \begin{cases} u^3 + v^3 + q = 0 \\ 3uv + p = 0. \end{cases} \]
From here we get
\[ \begin{cases} u^3 + v^3 = -q, \\ u^3v^3 = -p^2/27. \end{cases} \]
The \( t = u^3 \) and \( s = v^3 \) solve the quadratic equation
\[ w^2 + qw - p^3/27 = 0. \]
We get
\[ w = \frac{1}{2}(-q \pm \sqrt{\Delta}) \quad \text{where} \quad \Delta = q^2 + \frac{4p^3}{27}. \]
We get 

\[ u^3 = \frac{1}{2} (-q + \sqrt[3]{\Delta}) \]

\[ v^3 = \frac{1}{2} (-q - \sqrt[3]{\Delta}) \]

Then 

\[ u = \sqrt[3]{\frac{1}{2} (-q + \sqrt[3]{\Delta})} \]

\[ v = \sqrt[3]{\frac{1}{2} (-q - \sqrt[3]{\Delta})} \]

Note that there are three values of \( u \) and three values of \( v \) because a complex number has three third roots. Let us denote the three values of \( u \) as \( u_1, u_2, u_3 \), and the three values of \( v \) as \( v_1, v_2, v_3 \). The roots \( \tau = u + v \) of the original cubic equation are therefore

\[ u_1 + v_1, \quad u_2 + v_2, \quad u_3 + v_3, \]

\[ u_1 + v_2, \quad u_2 + v_3, \quad u_3 + v_1, \]

\[ u_1 + v_3, \quad u_2 + v_1, \quad u_3 + v_2. \]

Among these 9 pairs of \((u, v)\), we only take the ones such that \( uv = -\frac{b}{3} \).

One can check this relation by checking if \( \text{arg}(u) + \text{arg}(v) = \text{arg}(-\frac{b}{3}) \) in modulo \( 2\pi \). See the worksheet for an example.

If the equation is in a general form \( ax^3 + bx^2 + cx + d = 0 \), we can divide both sides by \( a \):

\[ \frac{x^3}{a} + \frac{b}{a} x^2 + \frac{c}{a} x + \frac{d}{a} = 0 \]

and then use the change of variable \( x = z + \frac{b}{3a} \). The equation that \( z \) satisfies will be of the form \( z^3 + px + q = 0 \).

Ex: 

\[ z^3 + 3z^2 + 1 = 0 \]

Change of variable \( z = \tau + \frac{3}{5} = \tau + 1 \), or equivalently \( \tau = z - 1 \).
Substitute this into the equation:

\[(2-1)^3 + 3(2-1)^2 + 1 = 0\]

or equivalently,

\[x^3 - 3x + 3 = 0\]

We know how to solve this equation by Cardano’s method. [See Problem 2 on the worksheet for example.]

\[x\]

The exponential function with complex variable

We know what \(e^x\) is when \(x \in \mathbb{R}\). That is

\[e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\]

We know what \(e^{iy}\) is when \(y \in \mathbb{R}\). That is

\[e^{iy} = 1 + \frac{iy}{1!} + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} + \cdots\]

\[= \cos y + i \sin y\]

How do we define \(e^z\) where \(z = x + iy\)?

There are two equivalent ways to define \(e^z\). The first way is

\[e^z = e^{x+iy} = e^x \cos y + ie^x \sin y\]

The second way is

\[e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots\]

Why are these two ways equivalent to each other? Let us put \(u = x\) and \(v = iy\). We know that

\[e^u = 1 + \frac{u}{1!} + \frac{u^2}{2!} + \frac{u^3}{3!} + \cdots\]

\[e^v = 1 + \frac{v}{1!} + \frac{v^2}{2!} + \frac{v^3}{3!} + \cdots\]

Then

\[e^u e^v = \left(1 + \frac{u}{1!} + \frac{u^2}{2!} + \frac{u^3}{3!} + \cdots\right) \left(1 + \frac{v}{1!} + \frac{v^2}{2!} + \frac{v^3}{3!} + \cdots\right)\]
\[ = 1 + (u + v) + \left( \frac{u^2}{2} + uv + \frac{v^2}{2} \right) + \ldots \]
\[ = 1 + \frac{uv}{1!} + \frac{(u + v)^2}{2!} + \ldots \]

Thus,
\[ e^x e^y = e^{x+y} = 1 + \frac{u+v}{1!} + \frac{(u+v)^2}{2!} + \ldots = 1 + \frac{u}{1!} + \frac{v}{2!} + \ldots \]

first def. of \( e^z \)

second def. of \( e^z \)

How do we define sine and cosine of complex variable? We know that \( e^{ix} = \cos x + i\sin x \) for any \( x \in \mathbb{R} \).

Now replace \( x \) by \(-x\):
\[ e^{-ix} = \cos(-x) + i\sin(-x) \]
\[ = \cos x - i\sin x. \]

From here we get:
\[ \cos x = \frac{e^{ix} + e^{-ix}}{2}, \quad \sin x = \frac{e^{ix} - e^{-ix}}{2i}. \]

These identities are true when \( x \) is a real number. We will now allow \( x \) to be any complex number. Thus,
\[ \cos z \overset{\text{def}}{=} \frac{e^{iz} + e^{-iz}}{2}, \quad \sin z \overset{\text{def}}{=} \frac{e^{iz} - e^{-iz}}{2i}. \]

These definitions are the same as
\[ \cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \ldots \]
\[ \sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \ldots \]

(One can check by writing the series for \( e^{iz} \) and \( e^{-iz} \).)

See computation examples on the worksheet.