Recall the definition of the exponential, sine and cosine functions:

\[ e^z = e^{x+iy} = e^x e^{iy} = e^x (\cos y + i\sin y) \]

or equivalently

\[ e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots \]

Sine function:

\[ \sin z = \frac{e^{iz} - e^{-iz}}{2i} \]

or equivalently

\[ \sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \ldots \]

Cosine function:

\[ \cos z = \frac{e^{iz} + e^{-iz}}{2} \]

or equivalently

\[ \cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \ldots \]

Other functions can also be defined from these functions, such as

\[ \tan z = \frac{\sin z}{\cos z} = \frac{1}{i} \frac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}} \]

\[ \cot z = \frac{\cos z}{\sin z} = i \frac{e^{iz} + e^{-iz}}{e^{iz} - e^{-iz}} \]

\[ \cosh z = \frac{e^z + e^{-z}}{2} \]

\[ \sinh z = \frac{e^z - e^{-z}}{2} \]

Let us investigate some geometric properties of the exponential function. The real-valued function \( f(x) = e^x \) can be represented by its graph (a curve). There is not much to say about this graph.
However, the complex-valued function \( f(z) = e^z \) has much richer geometric properties to discuss. The function \( f(z) = e^z \) maps each complex number to a complex number. Geometrically, \( f \) maps each point on the plane to another point on the plane. Thus, \( f \) is a geometric transformation on the plane.

One can ask: what is the image of a line, say \( x = 1 \), under the exponential map? One can use Mathematica to plot the line \( x = 1 \). First, we parametrize the line. There are two ways to write the equation of the line:

* real form: \[
\begin{align*}
  x &= 1 \\
  y &= t
\end{align*}
\]

* complex form: \( z = 1 + it \).

One can use the command `ParametricPlot` in Mathematica:

\[
\text{ParametricPlot[}\{1, t\}, \{t, -1, 1\}]\]

or equivalently

\[
\text{ParametricPlot[ReIm[1 + t \cdot \text{I}], \{t, -1, 1\}, AspectRatio \to \text{Automatic}, AxesOrigin \to \{0,0\}]}\]

To plot the image of this line under the exponential function, we only need to adjust the above command a little bit:

\[
\text{ParametricPlot[ReIm[Exp[1 + t \cdot \text{I}]], \{t, -1, 1\}, AspectRatio \to \text{Automatic}, AxesOrigin \to \{0,0\}]}\]
One can create a dynamic plot to better visualize the image of the line. See the instruction file Mapping properties of the exponential function.

We can observe from the picture that as $t$ travels from 0 to $2\pi$ ($\approx 6.28...$) the image of the line is a circle. We can also see that when $t$ reaches $2\pi$, we return to the original point. This hints a periodic behavior of the exponential function. Consider the equation

$$e^z = e^w.$$  

Write $z = a + ib$ and $w = c + id$. Then

$$e^z = e^a e^{ib} \quad \text{modulus is } e^a, \text{ argument is } b$$

$$e^w = e^c e^{id} \quad \text{modulus is } e^c, \text{ argument is } d$$

Thus, \[
\begin{cases}
e^a = e^c \\
b = d + k2\pi
\end{cases}
\]

In other words, $a = c$ and $b = d + k2\pi$.

We conclude that

$$e^z = e^w \iff \text{ and only if } z = w + i2k\pi \text{ for some } k \in \mathbb{Z}.$$  

This means $e^z$ is a periodic function with period $2\pi$.  

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We have seen how to sketch the image of a curve under the exponential map. Now let us sketch the image of a region, say the rectangle \([-1, 1] \times [0, 2\pi]\).

The rectangle can be parametrize as
\[
\begin{cases}
  x = t, & -1 \leq t \leq 1 \\
  y = s, & 0 \leq s \leq 2\pi
\end{cases}
\]

We can draw this rectangle on Mathematica using the command ParametricPlot.

```
ParametricPlot[ReIm[t + I*s], {t, -1, 1}, {s, 0, 2*Pi}]
```

To draw the image of this rectangle, we only need to adjust the previous command a little bit:

```
ParametricPlot[ReIm[Exp[t + I*s]], {t, -1, 1}, {s, 0, 2*Pi}, PlotRange -> Full]
```