Mapping properties of exponential function

In this note, we will use Mathematica to visualize some mapping properties of the exponential function \( f(z) = e^z \). The methodology explained below is applicable to any complex functions.

Each complex number \( z = x + iy \) corresponds to a point at position \((x, y)\) on the plane. Thus, \( f \) maps a point to a point. We say that \( f \) is a geometric transformation on the plane.

Each point \( A \) on the vertical line \( x = 1 \) gets mapped by \( f \) to another point (called the image of \( A \) under \( f \)) on the plane. It is natural to ask: what does the set of all those images as \( A \) varies on the line \( x = 1 \) look like?

The line \( x = 1 \) has complex parametrization \( z = 1 + it \) where \( t \in \mathbb{R} \). The plot of the line for \( t \in [0, 1] \) is as follows (Figure 1).

\[
\text{ParametricPlot}[[1, t], \{t, 0, 1\}]
\]
or equivalently

\[
\text{ParametricPlot}[[\text{ReIm}[1 + t*I], \{t, 0, 1\}]
\]

The command \texttt{ReIm} returns the list of real part and imaginary part. For example \texttt{ReIm[1+I*3]} returns \{1,3\}. To declare function \( f \), we use the command:

\[
f[z_\_] := \text{Exp}[z]
\]

To plot the image of the line under \( f \), we write (Figure 2)

\[
\text{ParametricPlot}[[\text{ReIm}[f[1 + t*I]], \{t, 0, 1\}, \text{AspectRatio} \to \text{Automatic},
\text{AxesOrigin} \to \{0, 0\}]
\]

The option \texttt{AspectRatio\to\text{Automatic}} is to make sure that the scales on the vertical and horizontal axes are the same. The option \texttt{AxesOrigin \to \{0,0\}} is to make sure that the vertical and horizontal axes meet each other at the origin \((0,0)\).

To get better visualization, we will plot in motion. That is, as we trace points on the line \( z = 1 + it \) from \( t = 0 \) to \( t = 1 \), we want to see how the images are drawn out. This can be done as follows. For each \( s > 0 \), we sketch the image of line \( z = 1 + it \) for \( t \in [0, s] \). Then allows \( s \) to increase from a small positive number, say 0.1, to 1 to see how the images are drawn out (Figure 3). We do so by using the command \texttt{Manipulate}.

\[
p[s_] :=
\text{ParametricPlot}[[\text{ReIm}[1 + t*I], \{t, 0, s\}, \text{AxesOrigin} \to \{0, 0\},
\text{PlotRange} \to \{0, 2\}, \{-1, 2\}]\]
\q[s_] :=
\text{ParametricPlot}[[\text{ReIm}[f[1 + t*I]], \{t, 0, s\}, \text{AxesOrigin} \to \{0, 0\},
\text{PlotRange} \to \{0, 3\}, \{0, 3\}]
\text{Manipulate}[[p[s], q[s]], \{s, .1, 1\}]
\]
The option \texttt{PlotRange \rightarrow \{(a,b),(c,d)\}} is to make sure that we view the graph only within the region $a \leq x \leq b$, $c \leq y \leq d$. Without specifying \texttt{PlotRange}, the viewing region may change as $s$ varies, which can cause annoyance.

In Figure 3, the plot on the left shows the line $x = 1$ as we draw it upward; the plot on the right shows the image of the line under the exponential function.

\textit{Can you adjust the above code to draw the image of the line $z = 1 + it$ as $t$ moves from 0 to 10? What does the shape look like?}