1. Find the region where function $f$ is differentiable. Where is it holomorphic? Find the derivative of $f$.

(a) $f(z) = x^2 + y^2 + i2xy$

We have

$u_x = 2x$  
$u_y = 2y$

$u_x = 2x$  
$u_y = -2y$

The points $z = x + iy$ where $f$ is differentiable are those where

\begin{align*}
  u_x &= u_y \\
  u_y &= -u_x
\end{align*}

This system becomes $y = 0$ and $x$ is arbitrary. Thus, $f$ is differentiable on the real line and not differentiable everywhere else. It is nowhere holomorphic.

$f(z) = u_x + iv_x = 2x + i2y = 2(x + iy) = 2z$.

(b) $f(z) = x^3 - 3xy^2 + i(3x^2y - y^3)$

$u_x = 3x^2 - 3y^2$  
$u_y = 3x^3 - 2y$

$v_x = -6xy$  
$v_y = 6x^2 y$

The system

\begin{align*}
  u_x &= u_y \\
  u_y &= -v_x
\end{align*}

becomes $-3y^2 = -2y$.

This gives $y = 0$ and $y = \frac{2}{3}$.

\begin{align*}
  u_x &= u_y \\
  u_y &= -v_x
\end{align*}

In conclusion, $f$ is differentiable on the lines $y = 0$ and $y = \frac{2}{3}$. It is nowhere holomorphic.

$f(z) = u_x + iv_x = (3x^3 - 3y) + i6xy$.
(c) \( f(z) = \text{Log } z. \)

Hint: Use Cauchy-Riemann equation and the fact that

\[
\text{Arg } z = \begin{cases} 
\arctan \left( \frac{y}{x} \right) & \text{if } x > 0, \\
\arccos \left( \frac{x}{\sqrt{x^2 + y^2}} \right) & \text{if } y > 0, \\
-\arccos \left( \frac{x}{\sqrt{x^2 + y^2}} \right) & \text{if } y < 0,
\end{cases}
\]

We know that \( f \) is discontinuous on the negative real line \( \mathbb{R} \leq 0 \), so it is not differentiable there.

\[
f(z) = \ln |z| + i \text{Arg } z
\]

\[
= \ln \sqrt{x^2 + y^2} + i \text{Arg } z
\]

\[
= \frac{1}{2} \ln(x^2 + y^2) + i \text{Arg } z.
\]

We have

\[
u_x = \frac{1}{2} \frac{2x}{x^2 + y^2} = \frac{x}{x^2 + y^2},
\]

\[
u_y = -\frac{1}{2} \frac{2y}{x^2 + y^2} = \frac{y}{x^2 + y^2}.
\]

In domain (1): \( x > 0 \).

\[
u(z, y) = \arctan \left( \frac{y}{x} \right) = \frac{-y/\sqrt{x^2 + y^2}}{1 + \left( \frac{y}{x} \right)^2} = -\frac{y}{x^2 + y^2}.
\]

\[
u_x = \frac{1}{1 + \left( \frac{y}{x} \right)^2} \cdot \frac{2x}{x^2 + y^2} = \frac{x}{x^2 + y^2},
\]

\[
u_y = \frac{1}{1 + \left( \frac{y}{x} \right)^2} \cdot \frac{-2y}{x^2 + y^2} = -\frac{y}{x^2 + y^2}.
\]

\[
\begin{cases} 
\text{Cauchy-Riemann eq. are satisfied for any } (x, y) \in (1). \\
\end{cases}
\]

Similarly, one can check that the Cauchy–Riemann eqs. are satisfied in (2) and (3).
Therefore, $f$ is differentiable anywhere in $\mathbb{C} \setminus \mathbb{R} \leq 0$. It is holomorphic on $\mathbb{C} \setminus \mathbb{R} \leq 0$.

\[ f'(z) = \frac{u_x + iu_y}{x^2 + y^2} = \frac{x}{x^2 + y^2} + i \frac{-y}{x^2 + y^2} = \frac{x - iy}{x^2 + y^2}. \]

One can further simplify this expression as follows:

\[ f'(z) = \frac{x - iy}{(x - iy)(x + iy)} = \frac{1}{x + iy} = \frac{1}{\bar{z}}. \]