For each function $f(z)$ given below, find an antiderivative $F(z)$. Determine the region where $F$ is holomorphic. Determine the region where $f$ is holomorphic.

(a) $f(z) = z\log z$

\[ I = \int z \log z \, dz \]

Let $u = \log z \quad dv = zdz$ \[ du = \frac{1}{z} \, dz \quad v = \frac{z^2}{2} \]

Then $du = \frac{1}{z} \, dz$ and $v = \frac{z^2}{2}$.

By Integration by part,

\[ I = uv - \int vdu = \frac{z^2}{2} \log z - \int \frac{z}{2} \frac{1}{z} \, dz \]

\[ = \frac{z^2}{2} \log z - \int \frac{z}{2} \, dz \]

\[ = \frac{z^2}{2} \log z - \frac{z^2}{4} + C \]

An antiderivative of $f(z)$ is

\[ F(z) = \frac{z^2}{2} \log z - \frac{z^2}{4} \]

$F$ is holomorphic on $\mathbb{C \setminus \{0\}}$. 
(b) \( f(z) = \frac{z}{z^2 + 1} \)

\[
I = \int \frac{z}{z^2 + 1} \, dz
\]

Let \( u = z^2 + 1 \). Then \( du = 2z \, dz \).

\[
I = \int \frac{1}{2} \frac{du}{u} = \frac{1}{2} \int \frac{1}{u} \, du
\]

There are many antiderivatives of \( \frac{1}{u} \). One of them is \( \log u \).

\[
I = \frac{1}{2} \log(u) = \frac{1}{2} \log(z^2 + 1).
\]

An antiderivative of \( f(z) \) is

\[
P(z) = \frac{1}{2} \log(z^2 + 1).
\]

In HW 5, we showed that \( \log(z^2 + 1) \) is continuous everywhere except on the line

\[
\begin{align*}
x &= 0 \quad \text{and} \quad y &= \pm 1.
\end{align*}
\]

Although \( f \) is continuous on \( \mathbb{C} \setminus \{\pm i\} \), it doesn't have an antiderivative on \( \mathbb{C} \setminus \{\pm i\} \).

Theorem: If \( G \) is a simply connected and \( f \) is holomorphic on \( G \) then \( f \) has an antiderivative on \( G \).

We will show this theorem later in the course.