Evaluate the following complex integrals:

(a) \( \int_{\gamma_1} \frac{1}{z} \, dz \) where \( \gamma_1 \) is the right half of the unit circle centered at the origin, starting at \( i \) and ending at \( -i \).

There are two methods to solve this problem.

**Method 1:** use parametrization of \( \gamma_1 \)

\[ \gamma(t) = e^{it} \text{ where } -\frac{\pi}{2} \leq t \leq \frac{\pi}{2} . \]

\[ \gamma'(t) = (e^{it})' = ie^{it} . \]

\[ \int_{\gamma_1} \frac{1}{z} \, dz = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{e^{it}} (ie^{it}) \, dt = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} i \, dt = i\pi . \]

**Method 2:** use Fundamental theorem of Calculus.

(See page 3)

(b) \( \int_{\gamma_2} \frac{1}{z} \, dz \) where \( \gamma_2 \) is the left half of the unit circle centered at the origin, starting at \( i \) and ending at \( -i \).

**Method 1:** use parametrization of \( \gamma_2 \)

\[ \gamma(t) = e^{it} \text{ where } \frac{\pi}{2} \leq t \leq \frac{3\pi}{2} . \]

\[ \gamma'(t) = (e^{it})' = ie^{it} . \]

\[ \int_{\gamma_2} \frac{1}{z} \, dz = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{1}{e^{it}} (ie^{it}) \, dt = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} i \, dt = i\pi . \]

**Method 2:** use Fundamental theorem of Calculus.

(See page 3)
(c) \[ \int_\gamma \frac{z}{z^2+1} \, dz \]
where \( \gamma \) is the straight line segment from the origin to \( 1+i \).

Put \( u = z^2+1 \).

Then \( du = 2z \, dz \).

\[
\int_\gamma \frac{z}{z^2+1} \, dz = \int_1^2 \frac{1}{u} \, du
\]

where \( \gamma \) is the image of the curve \( \gamma \) under the map \( z^2+1 \).

In other words,

\[
\eta(t) = \frac{t}{t^2+1} + 1 = t^2 + 2t^2 - t^2 + 1 = 1 + 2t^2
\]

We see that \( \eta \) is a line segment on the line \( z = 1 \).

The function \( f(u) = \frac{1}{u} \) has an antiderivative \( F(u) = \frac{1}{2} \log(u) \)
in the region \( G = \mathbb{C} \setminus \mathbb{R} \leq 0 \).

Because \( \gamma \) lies entirely in \( G \), we can use Fund. Thm. of Calc.

\[
\int_\gamma \frac{z}{z^2+1} \, dz = \int_1^2 \frac{1}{u} \, du = F(\eta(1)) - F(\eta(0)) = \frac{1}{2} \log(1+2t) - \frac{1}{2} \log(1) = \ldots
\]
(a) **Method 2**: use Fundamental Thm of Calculus.

\[ f(z) = \frac{1}{z} \] has antiderivative \( F(z) = \log z \) on the region \( G = \mathbb{C} \setminus \mathbb{R}_{<0} \).

We see that \( \gamma_1 \) lies entirely in \( G \).

By Fund. Thm. of Calc., we have

\[
\int_{\gamma_1} f(z) \, dz = F(-i) - F(i) = \log(-i) - \log(i) = -i \pi.
\]

(b) **Method 2**: use Fundamental Thm of Calculus.

\[ f(z) = \frac{1}{z} \] has antiderivative \( \tilde{F}(z) = \log(-z) \) on the region \( G = \mathbb{C} \setminus \mathbb{R}_{\geq 0} \).

We see that \( \gamma_2 \) lies entirely in \( G \).

By Fund. Thm. of Calc., we have

\[
\int_{\gamma_2} f(z) \, dz = \tilde{F}(-i) - \tilde{F}(i) = \log(i) - \log(-i) = i \pi.
\]