• Read the instruction of each problem carefully.

• The exam has 6 pages. **Circle your final results.** Provided at the bottom of this cover page are some helpful formulae.

• To get full credit for a problem **you must show your work.** Answers not supported by valid arguments will get little or no credit.

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Helpful trigonometric identities:

\[
\sin a \cos b = \frac{1}{2} \sin(a + b) + \frac{1}{2} \sin(a - b)
\]

\[
\sin a \sin b = \frac{1}{2} \cos(a - b) - \frac{1}{2} \cos(a + b)
\]

\[
\cos a \cos b = \frac{1}{2} \cos(a - b) + \frac{1}{2} \cos(a + b)
\]

\[
\sin^2 x = \frac{1 - \cos 2x}{2}, \quad \cos^2 x = \frac{1 + \cos 2x}{2}
\]

Derivative of inverse trigonometric functions:

\[
(arcsin x)' = \frac{1}{\sqrt{1 - x^2}}, \quad (arctan x)' = \frac{1}{1 + x^2}
\]
Problem 1. (10 pts) Express the following sum using \( \Sigma \) notation:

\[ 1 + 3 + 5 + 7 + \ldots + 99 \]

\[
\sum_{k=1}^{50} (2k-1) \quad \text{or} \quad \sum_{k=0}^{49} (2k+1) \\
\]

There are many other possible answers.

Problem 2. (10 pts) Evaluate the following sum:

\[
\sum_{k=1}^{5} (2k + 1) \\
\]

\[
= (2(1) + 1) + (2(2) + 1) + (2(3) + 1) + (2(4) + 1) + (2(5) + 1) \\
= 3 + 5 + 7 + 9 + 11 \\
= 35
\]
Problem 3. Consider the function \( f(x) = \sqrt{x} \) on the interval \([1, 2]\). Suppose one wants to write the midpoint Riemann sum of \( f \) on this interval with \( n = 5 \).

(a) (3 ts) Find the width of each subinterval.
\[
\frac{2 - 1}{5} = \frac{1}{5} = 0.2
\]

(b) (6 pts) Find the grid-points (the \( x_k \)'s).

\[
\begin{align*}
    x_0 &= 1 \\
    x_1 &= 1.2 \\
    x_2 &= 1.4 \\
    x_3 &= 1.6 \\
    x_4 &= 1.8 \\
    x_5 &= 2
\end{align*}
\]

(c) (6 pts) Find the sample points (the \( x_k^* \)'s).

\[
\begin{align*}
    x_0^* &= 1.1 \\
    x_1^* &= 1.3 \\
    x_2^* &= 1.5 \\
    x_3^* &= 1.7 \\
    x_4^* &= 1.9
\end{align*}
\]

(d) (5 pts) Write the midpoint Riemann sum of \( f \). The sum should not contain \( \Sigma \) sign nor the dots “…” It should contain only numbers. Do not evaluate the sum.
\[
\begin{align*}
    f(x_0^*) \cdot 0.2 + f(x_1^*) \cdot 0.2 + f(x_2^*) \cdot 0.2 + f(x_3^*) \cdot 0.2 + f(x_4^*) \cdot 0.2 \\
    \approx \sqrt{1.1} \cdot 0.2 + \sqrt{1.3} \cdot 0.2 + \sqrt{1.5} \cdot 0.2 + \sqrt{1.7} \cdot 0.2 + \sqrt{1.9} \cdot 0.2
\end{align*}
\]
Problem 4. Consider the curves $y = 3 - x^2$ and $y = x + 1$.

(a) (3 pts) Graph these curves on the same graph.

(b) (6 pts) Find the points of intersection of these curves.

\[3 - x^2 = x + 1 \iff x^2 + x - 2 = 0\]

two roots: $x = 1$ and $x = -2$

For $x = 1$: $y = 3 - (1)^2 = 2$

For $x = -2$: $y = 3 - (-2)^2 = -1$

Intercepts: $(1, 2)$ and $(-2, -1)$
(c) (6 pts) Shade the region enclosed by the curves. Then evaluate the area of this region.

\[
\int_{-2}^{1} (3 - x^2) - (x + 1) \, dx = \int_{-2}^{1} (2 - x - x) \, dx \\
= \left( 2x - \frac{x^3}{3} - \frac{x^2}{2} \right) \bigg|_{-2}^{1} \\
= \left( 2 - \frac{1}{3} - \frac{1}{2} \right) - \left( -4 + \frac{8}{3} - 2 \right) \\
= \frac{5}{2} - \frac{1}{2} + 6 - \frac{8}{3} = -1 - \frac{1}{2} + 6 = 5 - \frac{1}{2} = \frac{9}{2}
\]

**Problem 5.** Evaluate the following definite integrals:

(a) (10 pts)

\[
\int_{1}^{2} \frac{x^2 + 1}{x^3 + 3x} \, dx
\]

\[
u = x + 3x \\
du = (3x^2 + 3) \, dx \\
= 3(x^2 + 1) \, dx \\
\]

\[
\frac{x^2 + 1}{x^3 + 3x} \, dx = \frac{1}{3} \frac{du}{u} \\
\int_{1}^{2} \frac{1}{3} \frac{du}{u} = \frac{1}{3} \ln u \bigg|_{1}^{2} = \frac{1}{3} (\ln 2 - \ln 1) = \frac{1}{3} (\ln 2) 
\]

\[
\int_{1}^{4} \frac{1}{3} \frac{du}{u} = \frac{1}{3} \ln u \bigg|_{1}^{4} = \frac{1}{3} (\ln 4 - \ln 1) = \frac{1}{3} (\ln 4)
\]
(b) (10 pts)

\[ \int_0^2 \frac{3}{4 + x^2} \, dx \]

\[ u = \frac{x}{2} \quad x \bigg|_0^2 \begin{array}{c} \frac{3}{4} \end{array} \begin{array}{c} 1 \end{array} \]

\[ du = \frac{1}{2} \, dx \]

\[ \frac{3}{4 + \frac{x^2}{4}} \, 2 
\begin{array}{c} \frac{3}{4} 
\end{array} 2 
\begin{array}{c} \frac{1}{2} \end{array} \frac{1}{\sqrt{1 + \frac{u^2}{4}}} 
\begin{array}{c} \arctan \frac{u}{2} \bigg|_0^1 \end{array} 
\begin{array}{c} \frac{3.7}{8} \end{array} 
\]

(c) (10 pts)

\[ \int_0^{\pi/2} \sin^2 x \cos^3 x \, dx \]

\[ u = \sin x \quad x \bigg|_0^{\pi/2} \begin{array}{c} 0 \end{array} \begin{array}{c} 1 \end{array} \]

\[ du = \cos x \, dx \]

\[ \sin^2 x \cos^3 x \, dx = \frac{\sin x}{\cos x} \frac{\cos x}{1 - \sin^2 x} \, du = u^2 (1 - u^2) \, du \]

\[ \begin{array}{c} \frac{\pi}{4} 
\end{array} \begin{array}{c} \int \end{array} u^2 (1 - u^2) \, du = \int_0^1 (u^2 - u^4) \, du = \left( \frac{u^3}{3} - \frac{u^5}{5} \right) \bigg|_0^1 \]

\[ = \frac{1}{3} - \frac{1}{5} = \frac{2}{15} \]
Problem 6. (15 pts) Evaluate the length of the following curve

\[ y = \ln x - \frac{x^2}{8} \text{ on } [1, 2] \]

\[
\begin{align*}
 f'(x) &= \frac{1}{x} - \frac{x}{4} \\
 f''(x) &= \left(\frac{1}{x} - \frac{x}{4}\right)' = \frac{-1}{x^2} - \frac{1}{4} + \frac{x^2}{16} \\
 1 + f''(x)^2 &= \frac{1}{x^2} + \frac{1}{4} + \frac{x^2}{16} = \left(\frac{1}{x} + \frac{x}{4}\right)^2 \\
 \sqrt{1 + f''(x)^2} &= \frac{1}{x} + \frac{x}{4} \\
 L &= \int_1^2 \left(\frac{1}{x} + \frac{x}{4}\right) \, dx = \left. \left(\ln x + \frac{x^2}{8}\right) \right|_1^2 = \ln 2 + \frac{2^2}{8} - (\ln 1 + \frac{1^2}{8}) \\
 &= \ln 2 + \frac{3}{8}
\end{align*}
\]