Lecture 4 (1/16/2019)

- Trapezoid rule (didn’t have a chance to mention last lecture)
- Recall:

\[
\text{If } f : [a,b] \rightarrow \mathbb{R} \text{ is bounded and piecewise continuous on } (a,b),
\]

then \( f \) is Riemann integrable.

The Dirichlet’s function \( f : (0,1] \rightarrow \mathbb{R}, \quad f(x) = \begin{cases} 
1 & \text{if } x \text{ rational} \\
0 & \text{if } x \text{ irrational}
\end{cases} \)

is not Riemann integrable. Note that this function is nowhere continuous. (Why?)

Another example of non-integrable function:

\[
f(x) = \frac{1}{x}
\]

The function \( f(x) = \begin{cases} 
\frac{1}{x} & \text{if } x > 0 \\
0 & \text{if } x = 0
\end{cases} \)

is piecewise continuous, but not bounded.

Partition the interval \([0,1]\) into \( n \) equal subintervals.

\[
x_k = \frac{k}{n}
\]

Take sample point \( x_0^* \) on \([x_k, x_{k+1}]\) as follows:

\[
x_0^* = \frac{1}{n^2} \quad (\text{to make } f(x_0^*) \text{ big})
\]

\[
x_k^* = x_k = \frac{k}{n} \quad (\text{left point})
\]

Then the Riemann sum is

\[
I_n = f(x_0^*) \frac{1}{n} + f(x_1^*) \frac{1}{n} + f(x_2^*) \frac{1}{n} + \cdots + f(x_n^*) \frac{1}{n}
\]

\[
= \frac{1}{x_0^*} \frac{1}{n} + \frac{1}{x_1^*} \frac{1}{n} + \cdots + \frac{1}{x_n^*} \frac{1}{n}
\]

\[
= \left( \frac{1}{x_0^*} + \frac{1}{x_1^*} + \cdots + \frac{1}{x_n^*} \right) \frac{1}{n}
\]
\[
= \left( \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n}} + \cdots + \frac{1}{\sqrt{n}} \right) \frac{1}{n}
\]

\[
= \left( \frac{n}{1} + \frac{n}{2} + \frac{n}{3} + \cdots + \frac{n}{n-1} \right) \frac{1}{n}
\]

\[
= n \left( \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} \right)
\]

doesn't converge.

Thus: An unbounded function \( f : [a,b] \to \mathbb{R} \) is not \( R \)-integrable.

Recall:

Definite integral is additive:

\[
\int_{a}^{b} (f(x) + g(x)) \, dx = \int_{a}^{b} f(x) \, dx + \int_{a}^{b} g(x) \, dx
\]

If \( f \) and \( g \) are integrable on \([a,b]\) then so is \( f + g \).

Definite integral is scaling multiplicative:

\[
\int_{a}^{b} c f(x) \, dx = c \int_{a}^{b} f(x) \, dx
\]

If \( f \) is Riemann integrable then so is \( cf \).
\[ \int_{a}^{b} f(x) \, dx \ldots \text{depends only on the function } f, \text{ and interval } [a, b]. \]

How to compute \( \int_{a}^{b} x \, dx \) without resorting to Riemann sums?

Put \( F(y) = \int_{a}^{y} f(x) \, dx \)

\[ F(y+h) - F(y) = \text{area of the thin strip} \]

\[ \approx f(y) \frac{h}{\text{height}} \times \text{width} \]

Approximation gets better as \( h \) shrinks to 0.

\[ \frac{F(y+h) - F(y)}{h} \approx f(y) \]

In fact, \( \lim_{h \to 0} \frac{F(y+h) - F(y)}{h} = f(y) \)

In other words, \( F'(y) = f(y) \)

\( F \) is an antiderivative of \( f \).

**Def:** A function \( g : (a, b) \to \mathbb{R} \) is called an antiderivative of \( f \) if \( g'(x) = f(x) \) for all \( x \in (a, b) \).

**Ex.**

\( f(x) = x \) has antiderivative \( \frac{x^2}{2} \)

\( f(x) = x^2 \) has antiderivative \( \frac{x^3}{3} \)

\( f(x) = x^n \) has antiderivative \( \frac{x^{n+1}}{n+1} \) \( (n \neq -1) \)

Antiderivative is not unique, for example \( \frac{x^2}{2} + 1 \) is another antiderivative of \( f(x) = x \).

**Thm:** Antiderivatives are unique up to a constant.
Why? Suppose both \( g(x) \) and \( h(x) \) are antiderivatives of \( f(x) \) on \((a,b)\).

\[
g'(x) = h'(x) = f(x)
\]

Put \( k(x) = g(x) - h(x) \). Then \( k'(x) = g'(x) - h'(x) = 0 \). We claim that \( k \) must be a constant function.

Take two points \( x_1, x_2 \) between \( a \) and \( b \).

By mean value theorem, there exists \( c \in (x_1, x_2) \) such that

\[
\frac{k(x_1) - k(x_2)}{x_1 - x_2} = k'(c)
\]

This is 0

Therefore, \( k(x) = k(x_2) \).

By:

All antiderivatives of \( f(x) = x \) are \( \frac{x^2}{2} + C \)

\[
\int f(x) \, dx = \frac{x^2}{2} + C
\]

The notation \( \int f(x) \, dx \) represents all antiderivatives of \( f \).

Ex.:

\[
\int x \, dx = \frac{x^2}{2} + C
\]

\[
\int x^2 \, dx = \frac{x^3}{3} + C
\]

\[
\int \cos x \, dx = \sin x + C
\]

called constant of integration