Lecture 5 (1/23/2019)

\[ F(y) = \int_a^y f(x) \, dx \]

Recall: \( F(y) = f(y) \)

Rename: \( F'(x) = f(x) \)

\( F \) is an antiderivative of \( f \).

Note:

\[ \lim_{n \to \infty} \sum_{k=0}^{n-1} f(x_k) \Delta x \]

is denoted by \( \int_a^b f(x) \, dx \)

\( x \) here is called independent variable. Its name is not important. One can write

\[ \int_a^b f(x) \, dx = \int_a^b f(y) \, dy = \int_a^b f(t) \, dt = \ldots \]

Any other antiderivative of \( f \) must have the form \( G(x) = F(x) + C \)

where \( C \) is a constant (called constant of integration).

Ex:

Find \( \int_0^1 x^2 \, dx \)

For convenience, we pick a different name for the independent variable under the integral sign:

\[ \int_0^1 t^2 \, dt \]

Put \( F(x) = \int_0^x t^2 \, dt \). We know that \( F'(x) = x^2 \).

Thus, \( F(x) = \frac{x^3}{3} + C \). How to find \( C \)?

We see that \( F(0) = \int_0^0 t^2 \, dt = 0 \).
This implies $c = 0$. Then $F(x) = \frac{x^3}{3}$.

Then $\int_0^1 x^2 \,dx = F(1) = \frac{1}{3}$.

* Indefinite integral:

The notation $\int f(x) \,dx$ represents all antiderivatives of $f$.

E.g.:

$\int x \,dx = \frac{x^2}{2} + C$

$\int e^x \,dx = e^x + C$

$\int \sin x \,dx = -\cos x + C$

* Theorem: (Fundamental theorem of calculus)

Let $G$ be an antiderivative of $f$. Then

$\int_a^b f(\alpha) \,d\alpha = G(b) - G(a)$

Why? $G(x) = F(x) + C$, where $F(x) = \int_a^x f(t) \,dt$

Then

$G(b) = F(b) + C$

$G(a) = F(a) + C = C$

Then

$G(b) - G(a) = F(b) = \int_a^b f(t) \,dt$.

The value of this theorem can be seen in different aspects:

* Provide an analytical method to compute areas under curves.

  (when we know an antiderivative of a function, we know the area)

* Connection between two concepts: derivative and integral.
\[ \int_a^b G'(t) \, dt = G(b) - G(a) \]

Note that this relation is not trivial from the definition of derivative and integral:
- derivative as limit of difference quotient
- (Definite) integral as limit of Riemann sum.

The difference \( G(b) - G(a) \) is often abbreviated as \( G(x) \bigg|_a^b \).

Example:
\[ \int_0^4 x^3 \, dx = \frac{x^4}{4} \bigg|_0^4 = \frac{4^4}{4} - \frac{0^4}{4} = \frac{256}{4} = 64 \]
\[ \int_0^\pi \sin(x) \, dx = \left[ -\cos(x) \right]_0^\pi = (-\cos(\pi)) - (-\cos(0)) = 1 - (-1) = 2 \]

Area between curves

The (unsigned) area of the region enclosed between the curves \( x = -1, \ x = 2, \ y = x^3, \ y = 0 \) is
\[ \int_{-1}^2 |x^3| \, dx = -\int_{-1}^0 x^3 \, dx + \int_0^2 x^3 \, dx \]
\[ = \left[ -\frac{x^4}{4} \right]_0^2 + \left[ \frac{x^4}{4} \right]_0^2 \]
\[ = \frac{16}{4} - \frac{1}{4} = \frac{15}{4} \]

In general, the area of the region between the curve \( x = a, \ x = b, \ y = f(x), \ y = 0 \) is
\[ \int_a^b |f(x)| \, dx \]

The area of the region between \( x = a, \ x = b, \ y = f(x), \ y = g(x) \) is
\[ \int_a^b |f(x) - g(x)| \, dx \]
\[
\begin{align*}
\text{Region between } x = 0, x = 2, \\
y = x^2, y = 2x
\end{align*}
\]
has area
\[
\int_0^2 (2x-x^2) \, dx = \left[ x^2 - \frac{x^3}{3} \right]_0^2
\]
\[
= 2 - \frac{8}{3} = \frac{4}{3}
\]

* Length of an arc (revisited):*
\[
\begin{align*}
l &= \lim_{n \to \infty} \sum_{k=0}^{n-1} \sqrt{(x_{k+1} - x_k)^2 + (f(x_{k+1}) - f(x_k))^2} \\
\text{This sum doesn't look like a Riemann sum, but we claim that it is a Riemann sum of the function } \sqrt{1 + f(x)^2}.
\end{align*}
\]

How to see this? Hint:
\[
\begin{align*}
\sum_{k=0}^{n-1} \sqrt{(x_{k+1} - x_k)^2 + (f(x_{k+1}) - f(x_k))^2} &= \sum_{k=0}^{n-1} \sqrt{1 + \left( \frac{f(x_{k+1}) - f(x_k)}{x_{k+1} - x_k} \right)^2 (x_{k+1} - x_k)} \\
\text{(continue next time)}
\end{align*}
\]