Recall:

Area = \int_a^b |f(x)| \, dx

Signed area = \int_a^b f(x) \, dx

Length = \int_a^b \sqrt{1 + f'(x)^2} \, dx

How about average?

One knows how to compute average of a discrete function. For example,

\[ f(M) = \text{# calories you consume on Monday} \]
\[ f(T) = \text{# calories you consume on Tuesday} \]

Average of calorie consumption per day = \frac{f(M) + f(T) + \ldots + f(S)}{f}

Suppose you wear a Fitbit watch that tells you how much energy your body spends at each time.

\[ f(x) \]

Continuous variable

What is the average energy spent from 8 AM to 8:30 AM?

* Average of \( f \) over interval \([a, b] \):

Recall:

\[ \sum_{k=0}^{n-1} f(x_k) \]

\[ \int_a^b f(x) \, dx = \lim_{n \to \infty} \frac{b-a}{n} \sum_{k=0}^{n-1} f(x_k) \]

Divide both sides by \( b-a \):
\[
\frac{1}{b-a} \int_a^b f(x) \, dx = \lim_{n \to \infty} \sum_{k=0}^{n-1} f(x_k) \frac{1}{n} = \lim_{n \to \infty} \frac{f(x_0)+f(x_1)+\ldots+f(x_{n-1})}{n}
\]

This quantity can be reasonably defined as the average of \( f \) over the interval \([a,b]\).

\[\text{Ex: } f(x) = x^3 \text{ on } [0,2]\]

\[
\text{Avg}(f) = \frac{1}{2-0} \int_0^2 x^2 \, dx = \frac{1}{2} \frac{x^4}{4} \bigg|_0^2 = 2
\]

**Velocity, acceleration, displacement:**

\[
s'(t) = a(t)
\]

\[
v'(t) = v(t)
\]

\[
s(t) : \text{position function}
\]

\[
v(t) : \text{(instant) velocity}
\]

\[
a(t) : \text{acceleration}
\]

\[
s(b) - s(a) = \int_a^b v'(t) \, dt \quad \text{displacement between } t=a \text{ and } t=b
\]

\[
\int_a^b |v(t)| \, dt \quad \text{distance travelled between } t=a \text{ and } t=b
\]

**Ex:** Suppose you drive with a SnapChat device in your car.

At the end of the trip, the device gives you a graph of your speed as a function of time.

Suppose your car's velocity profile is

\[
v(t) = \begin{cases} 3t & 0 \leq t < 20 \\ 60 & 20 \leq t \leq 45 \\ 240 - 4t & t > 45 \end{cases}
\]

What is the average speed in the first 70 s?
\[ \frac{1}{70} \int_{0}^{70} v(t) \, dt = \frac{1}{70} \left( \int_{0}^{20} v(t) \, dt + \int_{20}^{45} v(t) \, dt + \int_{45}^{70} v(t) \, dt \right) \]

\[ I_1 = \int_{0}^{20} v(t) \, dt = \frac{v(t)}{2 \cdot 0} \bigg|_{0}^{20} = 600 \]

\[ I_2 = \int_{20}^{45} 60 \, dt = 60 \cdot t \bigg|_{20}^{45} = 1500 \]

\[ I_3 = \int_{45}^{70} (240-4t) \, dt + \int_{60}^{70} (4t-240) \, dt \]

\[ I_4, I_5 \]

Average speed = \[ \frac{1}{70} (I_1 + I_2 + I_3 + I_4) = \ldots \]

+ Integration by substitution:

Compare differentiation (how to compute derivatives) and integration (how to compute integrals)

**Differentiation**

- Sum: \( (f+g)' = f' + g' \)
- Scaling: \( (cf)' = cf' \) for constant \( c \)
- Product: \( (fg)' = f'g + fg' \)
- Chain rule: \( (f(g(x)))' = f'(g(x))g'(x) \)

**Integration**

- \( \int (f(x) + g(x)) \, dx = \int f(x) \, dx + \int g(x) \, dx \)
- \( \int cf(x) \, dx = c \int f(x) \, dx \)

Integration by parts

Integration by substitution

Integration by substitution can be summarized in one line:

\[ \int f(u) \, du = \int f(u(x)) u'(x) \, dx \]
Why is this true?

Let F be an antiderivative of f. 

Integrate the equation \( [F(g(x))]' = f'(g(x))g'(x) \) with respect to \( x \):

\[
\int [F(g(x))]' \, dx = \int f'(g(x))g'(x) \, dx
\]

In other words,

\[
\int f(g(x))g'(x) \, dx = F(g(x)) + C \quad (**)
\]

In practice, this formula amounts to a change of variable

\[
u = g(x)
\]

\text{new var.} \quad \text{old var.}

Formally, \( du = g'(x)dx \) (notation from differential calculus)

Thus, one can rewrite (**) as

\[
\int f(g(x))g'(x) \, dx = F(u) = \int f(u) \, du
\]

In practice, the formula (**) is used as

\[
\int \frac{h(u(x))u'(x)}{f(u)} \, dx = \int h(u) \, du
\]

Procedure:

1) Identify a new variable \( u = u(x) \) (hidden in \( f(u) \))
2) Compute \( du = u'(x) \, dx \)
3) Change \( f(u) dx \) into an expression of \( u \) only, say \( h(u) du \)
   If this can't be done, the new variable is not a good choice.
4) Integrate \( \int h(u) \, du \)

The result is a function of \( u \) (plus constant \( C \))
5) Substitute back \( u = u(x) \).
\[ \text{Ex:} \quad \int \frac{x}{x^2 + 1} \, dx \]

Put \( u = x^2 + 1 \). Then \( du = 2x \, dx \).

\[ f(x) \, dx = \frac{x}{x^2 + 1} \, dx = \frac{x \, dx}{u} = \frac{1}{2} \, du = \frac{1}{2u} \, du \]

Now integrate both sides:

\[ \int f(x) \, dx = \int \frac{1}{2u} \, du = \frac{1}{2} \ln |u| + C \]

\[ = \frac{1}{2} \ln (x^2 + 1) + C \]