Inverse trigonometric functions:

\[ \sin \theta = x \quad \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}] \]

There are infinitely many such \( \theta \)’s. However, there is only one \( \theta \) in the interval \( \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \).

\[ \arcsin : [-1, 1] \rightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \text{ is the inverse of the } \sin \text{ function.} \]

\[ \arcsin x = \theta \quad \Rightarrow \quad \sin \theta = x \]

What is the derivative of \( \arcsin \) ?

\[ (\arcsin x)' = \frac{1}{\sqrt{1-x^2}} \] (a consequence of the chain rule)

One can define the inverse of the tangent function likewise:

\[ \tan \theta = x \in \mathbb{R} \]

There are infinitely many such \( \theta \)’s.

\[ \arctan : \mathbb{R} \rightarrow \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \]

is the inverse function of \( \tan : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \rightarrow \mathbb{R} \)

\[ y = \tan x \quad \text{take reflection about the line } y = x \]

\[ y = \arctan x \]
\[(\arctan u)' = \frac{1}{1+u^2}\]

**Ex:** Compute 
\[\int_0^1 \frac{1}{1 + x^2} \, dx\]

\[= \arctan x \bigg|_0^1 = \frac{\pi}{4} - 0 = \frac{\pi}{4}\]

**Ex:** 
Compute 
\[\int_0^3 \frac{1}{4 + 9x^2} \, dx\]

\[I = \frac{1}{4} \int_0^3 \frac{1}{1 + \frac{9x^2}{4}} \, dx\]

Let \(u = \frac{3x}{2}\), \(du = \frac{3}{2} \, dx\)

\[I = \frac{1}{4} \int_{3/2}^{9/4} \frac{1}{1 + u^2} \frac{2}{3} \, du\]

\[= \frac{1}{6} \arctan u \bigg|_{3/2}^{5/4}\]

\[= \frac{1}{6} \left( \arctan \left( \frac{5}{4} \right) - \arctan \left( \frac{3}{2} \right) \right)\]

**Ex:**

\[I = \int_0^1 \frac{1}{\sqrt{4-x^2}} \, dx\]

Let \(u = \frac{x}{2}\), \(du = \frac{1}{2} \, dx\)

\[\frac{1}{\sqrt{4-u^2}} \, du = \frac{1}{\sqrt{4-4u^2}} \, 2 \, du = \frac{1}{\sqrt{1-u^2}} \, du\]
\[ I = \int_0^1 \frac{1}{\sqrt{1-u^2}} \, du = \arcsin u \bigg|_0^1 = \arcsin \frac{1}{2} - \arcsin 0 = \frac{\pi}{6} \]

*Compute volume by slicing:*

Consider a solid that extends from \( x = a \) and \( x = b \). Suppose the cross section area at position \( x \) is \( A(x) \).

Volume is approximated by

\[ \sum A(x) \Delta x \]

This is a Riemann sum of the function \( A(x) \) on the interval \([a, b]\).

Thus, the exact volume is

\[ V = \int_a^b A(x) \, dx \]

Ex:

Find volume of the cone with height \( h \) and circular base of radius \( R \).

Cross section at position \( x \) is a circle with radius \( r \).

\[ \tan \theta = \frac{R}{h} = \frac{r}{h-x} \]

Thus,

\[ r = \frac{R}{h} (h-x) \]

\[ A(x) = \pi r^2 = \frac{\pi R^2}{h^2} (h-x)^2 \]
Volume of the cone = \( \int_0^h A(x) \, dx = \frac{\pi R^2}{h^2} \int_0^h (h-x)^2 \, dx \)

\[ = \frac{\pi R^2}{h^2} \left[ -\frac{(h-x)^3}{3} \right]_0^h \]

\[ = \frac{\pi R^2}{h^2} \frac{h^3}{3} \]

\[ = \frac{\pi R^2 h}{3} \]