Homework 2

1. Check if each following map is a linear map. If it is, explain why (by verifying the 2 criteria). If it is not, show how one of these criteria is violated.
   
   (a) \( f : \mathbb{R}^2 \to \mathbb{R}^2, f(x_1, x_2) = (x_1 + x_2, x_1 - x_2). \)
   
   (b) \( f : \mathbb{R}^2 \to \mathbb{R}, f(x_1, x_2) = x_1^2 + x_2^2. \)
   
   (c) \( f : \mathbb{R}^3 \to \mathbb{R}, f(x_1, x_2, x_3) = x_1x_2x_3. \)

2. Let
   
   \[
   A = \begin{bmatrix}
   2 & -1 & -1 \\
   1 & 0 & 3 \\
   -3 & 1 & -2
   \end{bmatrix}, \quad
   B = \begin{bmatrix}
   1 & 2 & 0 \\
   -2 & 3 & 1 \\
   -1 & 4 & -3
   \end{bmatrix}
   \]

   Compute \((2A - B)^2\).

3. Recall that zero matrix is a matrix whose every entry is equal to 0. For convenience, an \(m \times n\) zero matrix is often denoted as 0 (as if it were the number zero). The size of the matrix is usually understood in the context.

   Give an example of a 2-by-2 nonzero matrix \( A \) such that \( A^2 = 0 \).

4. Determine (i.e. write the formula of) a linear map \( f : \mathbb{R}^2 \to \mathbb{R} \) such that \( f(1, 2) = 1 \) and \( f(2, 5) = 4 \).

5. Let
   
   \[
   A = \begin{bmatrix}
   2 & 0 & 1 & 3 \\
   1 & -3 & 4 & 0 \\
   -1 & -4 & 3 & -2
   \end{bmatrix}
   \]

   Find the linear map associated with \( A \). (This includes finding the domain, the target set, and an explicit formula of \( f \)).

6. Find the matrix associated with the following linear map:
   
   (a) \( f : \mathbb{R}^2 \to \mathbb{R}^2, f(x_1, x_2) = (2x_1 - x_2, 0). \)
   
   (b) \( f : \mathbb{R}^3 \to \mathbb{R}^2, f(x_1, x_2, x_3) = (x_1 - x_2 - x_3, x_2). \)
   
   (c) \( f : \mathbb{R} \to \mathbb{R}^2, f(x) = (2x, -x). \)

7. Let \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) and \( g : \mathbb{R}^2 \to \mathbb{R}^3 \) be linear maps given by
   
   \[
   f : \mathbb{R}^2 \to \mathbb{R}^2, \quad f(x_1, x_2) = (2x_2, x_1),
   \]
   
   \[
   g : \mathbb{R}^2 \to \mathbb{R}^3, \quad g(x_1, x_2) = (x_2, x_1 - x_2, x_1),
   \]
   
   \[
   h : \mathbb{R}^2 \to \mathbb{R}^2, \quad h(x_1, x_2) = (0, 3x_2 - 2x_1).
   \]

   What are the matrices associated with \( f, g \) and \( h \)? To each of the following maps, first write an explicit formula, then find the associated matrix:
   
   (i) \( f + h \)
   
   (ii) \( f - 2h \)
   
   (iii) \( g \circ f \)

8. Do Problems 1, 5, 7 of Section 3.8 (page 53) of the textbook by using Gauss elimination method.