Homework 4

1. This problem is an application of inverse matrix: If a system of linear equations has the same number of equations as unknowns and if the coefficient matrix is invertible, then the system can be solved by inverse matrix. Consider the system:

\[
\begin{align*}
2y - z &= -2 \\
5x + 2y + 3z &= 4 \\
7x + 3y + 4z &= -5
\end{align*}
\]

(i) Write the system in matrix form \(AX = b\).
(ii) Find \(\det A\). Is \(A\) invertible?
(iii) Find \(A^{-1}\).
(iv) Use the formula \(X = A^{-1}b\) to find solutions to the system.

2. Check if each of the following statements is true for all 2-by-2 matrices \(A\) and \(B\). If it is, justify your answer. If it is not, give a counterexample.

(a) \((A + B)^{-1} = A^{-1} + B^{-1}\) (assuming \(A\) and \(B\) are invertible)
(b) \((AB)^{-1} = B^{-1}A^{-1}\) (assuming \(A\) and \(B\) are invertible)
(c) \(AA^T = A^TA\) (Recall that \(A^T\) denotes the transpose matrix.)

3. Find the determinant of the following matrices.

(a) \[
\begin{bmatrix}
1 & 2 \\
3 & 4
\end{bmatrix}
\]
(b) \[
\begin{bmatrix}
2 & 3 \\
3 & 2
\end{bmatrix}
\]
(c) \[
\begin{bmatrix}
1 & 1 & 2 \\
2 & 3 & 1 \\
3 & 4 & -5
\end{bmatrix}
\]
(d) \[
\begin{bmatrix}
2 & 3 & 1 \\
-1 & 2 & 3 \\
3 & 2 & -1
\end{bmatrix}
\]

4. Find all values \(c\) such that the following matrix is invertible

\[
\begin{bmatrix}
1 & c & 0 \\
c & 1 & 0 \\
0 & 1 & c
\end{bmatrix}
\]

5. Let

\[
A = \begin{bmatrix}
2 & 1 \\
3 & 4
\end{bmatrix}
\]

(a) Determine all numbers \(\lambda\) such that the matrix \(A - \lambda I_2\) fails to be invertible.
(b) Find a nonzero vector \(v\) such that \(Av = 5v\). (Hint: write \(5v\) as \(5I_2v\).)

6. (Problems 2, 5, 6, 8 of Section 5.4 on page 76 textbook) Determine if the given vectors are linearly dependent or linearly independent. If they are linearly dependent, find a nontrivial linear combination of the vectors that has sum 0.
(a) \( v_1 = (3, 2), v_2 = (6, 4) \).
(b) \( v_1 = (-1, 3, -2), v_2 = (3, 1, 0), v_3 = (2, -1, 1) \).
(c) \( v_1 = (2, 1, 0), v_2 = (0, 1, 0), v_3 = (-1, 2, 0) \).
(d) \( v_1 = (1, 1, 0, 1), v_2 = (0, 1, 1, 0), v_3 = (4, 2, -2, 4) \).

7. Determine (i.e. write an explicit formula for) a linear map \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) such that \( f(2, 1) = (-1, 3) \) and \( f(3, 2) = (2, 0) \). (Hint: find the matrix associated with \( f \)).

8. Let \( L \) be the line given by the equation \( 3x - 4y = 0 \) on the plane. Let \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) be the (orthogonal) projection onto \( L \). That is, \( f \) maps each point on the plane to its projection on \( L \). Recall that a point can be thought as a vector based at the origin, and vice versa. One can write a formula for \( f \) by following the below steps:

   (i) Draw the line \( L \). (Can you convince yourself using picture that \( f \) should be a linear map? For example, draw any two vectors on the plane. Is the projection of the sum equal to the sum of the projections? Similar question for scaling.)
   (ii) Find a unit direction vector of \( L \) (i.e. a vector that has length 1 and is parallel to \( L \)).
   (iii) Determine the projections of vectors \( e_1 = (1, 0) \) and \( e_2 = (0, 1) \) onto \( L \). (Hint: use dot product.)
   (iv) What are \( f(e_1) \) and \( f(e_2) \)? Determine the matrix \( A \) associated with \( f \).
   (v) Write an explicit formula for \( f \) (i.e. \( f(x_1, x_2) = \ldots \))