Homework 6

1. This exercise is an example of how find the lowest order of Taylor polynomial to compute $e^2 \sin 2$ with error of at most $10^{-2}$. Put

$$f(x) = e^x \sin \left( \frac{x}{2} \right)$$

(a) Compute $f'(x)$ and $f''(x)$.
(b) Find the Taylor polynomials $T_0$, $T_1$, $T_2$ of $f$ about the base point 0.
(c) From Part (a), verify that $|f'(x)| \leq e^{x/2}$ and $|f''(x)| \leq e^{x/2}$.
(d) What does Lagrange’s theorem say about the error term $R_n(4) = f(4) - T_n(4)$?
(e) It is know that (you don’t have to verify) $|f^{(n)}(x)| \leq e^{x/2}$ for any $x \in \mathbb{R}$. Find an upper bound in terms of $n$ for the error term $R_n(4)$. (Hint: use $e^2 < 8$).
(f) With the upper bound found in Part (e), use your calculator to find $n$ (by trying $n = 1, 2, 3, \ldots$) such that $R_n(4) < 10^{-2}$.

2. Consider the series

$$1 - \frac{1}{3^2} + \frac{1}{3^4} - \frac{1}{3^6} + \frac{1}{3^8} + \ldots$$

(a) Express this series using $\Sigma$ notation.
(b) Let $S_n$ be the $n$'th partial sum, which is the sum of the first $n$ terms. Find a general formula of $S_n$ in terms of $n$. Hint: write $(-1/3)S_n$, then subtract $S_n$ from it.
(c) Find $\lim_{n \to \infty} S_n$.
(d) Does the series converge? If it does, what is the sum of the series?

3. Consider the series

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{5}} + \ldots$$

(a) Express this series using $\Sigma$ notation.
(b) Verify that

$$\sqrt{k+1} - \sqrt{k} = \frac{1}{\sqrt{k+1} + \sqrt{k}}$$

(c) Based on the above formula, verify that for every $k \geq 1$

$$\sqrt{k+1} - \sqrt{k} < \frac{1}{2\sqrt{k}}$$

(d) Let $S_n$ be the $n$'th partial sum, which is the sum of the first $n$ terms. Verify that $S_n > 2(\sqrt{n+1} - 1)$.
(e) Find $\lim_{n \to \infty} S_n$.
(f) Does the series converge or diverge?