Lecture 1 (1/7/2019)

How to compute the values of a map?

Given a map $f: \mathbb{R} \to \mathbb{R}$, one can describe $f$ by its graph.

- **Linear map**: $f(x) = ax$  
- **Affine map**: $f(x) = ax + b$  
- **Nonlinear map**

Linear map: map whose graph is a line passing through the origin. Linear map is completely determined by number $a$, the slope of the line.

- $f(x) = ax$: one multiplication
- If we know $f(1)$, we know $f(x)$ for every $x$.
- $f(x) = ax^2 + bx + c$: 3 multiplications, 2 additions
- If we know $f(0)$, $f'(0)$, $f''(0)$, can determine $a$, $b$, $c$, and thus $f(x)$.

How about $f(x) = \sin x$ or $e^x$? How can a calculator compute $e^{x^2}$ based on additions and multiplications?

We need to approximate. To obtain exact computation, addition and multiplication are not enough. We need calculus.
A nonlinear map is more difficult to calculate. Suppose one knows everything about \( f \) at point \( x_0 \). How to compute \( f(x) \)?

If \( x \) is time variable, this is the problem of predicting the future given all necessary information at the initial time.

\[
\begin{align*}
\text{If } x &\approx x_0, \quad f(x) \approx f(x_0) \\
\text{If } x \text{ is further away,} & \\
& f(x) \approx f(x_0) + f'(x_0)(x-x_0) + f''(x_0)(x-x_0)^2 \\
& \quad + \frac{f'''(x_0)}{3!}(x-x_0)^3 + \cdots
\end{align*}
\]

\[
\lim_{x \to x_0} \frac{h(x)}{(x-x_0)^n} = \lim_{x \to x_0} \frac{h'(x)}{n!} = \lim_{x \to x_0} \frac{h''(x)}{2n!} = \frac{h''(x_0)}{2n!} = \frac{h''(x_0)}{2}
\]

Thus,

\[
\frac{h(x)}{(x-x_0)^n} \approx \frac{h''(x_0)}{2n!}
\]

which leads to (4).

For a large class of functions \( f \),

\[
f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \frac{f'''(x_0)}{3!}(x-x_0)^3 + \cdots
\]

(\text{infinite sum})
We will make rigorous the definition of this sum later in the course. Such a sum is called series (in this case, Taylor series of function $f$).

Linear map is roughly speaking the 1st order approximation of a nonlinear map. The strength of linear map is more clear in higher dimension.

Example: $f: \mathbb{R}^2 \rightarrow \mathbb{R}$

\[ f(x, y) = 2x + 3y \quad \text{linear map} \]

\[ f(1, 0) = 2 \quad f(2, 3) = (2, 3) \cdot \mathbf{v} \]

\[ f(0, 1) = 3 \]

\[ f(v) = x f(e_1) + y f(e_2) \]

Knowing $f$ at $e_1$ and $e_2$, one can find $f(v)$ for any vector $v$.

The calculation of linear maps becomes algebra.

In the first half of the course, we'll study the algebra of linear map.

\[ f(v) = A \mathbf{v} \]

Matrix $\rightarrow$ graph of a map

In the second half, we'll study the calculus of nonlinear map.

Series $\rightarrow$ approximation
+ Review on vectors:

\[ \mathbb{R}^n : \mathbf{x} = (x_1, x_2, \ldots, x_n) \text{ : vector of } n \text{ components} \]

\[ \mathbf{z} = (z_1, z_2, \ldots, z_n) \]

\[ \mathbf{y} = (y_1, y_2, \ldots, y_n) \]

**Sum:**

\[ \mathbf{x} + \mathbf{y} = (x_1 + y_1, \ldots, x_n + y_n) \]

**Scalar mult.:** \((c \in \mathbb{R})\)

\[ c\mathbf{z} = (cz_1, cz_2, \ldots, cz_n) \]

**Dot product:**

\[ \mathbf{x} \cdot \mathbf{y} = x_1y_1 + \ldots + x_ny_n \]

**Length:**

\[ |\mathbf{x}| = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2} \]

\[ \mathbf{x} \cdot \mathbf{y} = |\mathbf{x}| |\mathbf{y}| \cos \theta \]

Functions on \(\mathbb{R}^2\):

\[ f : \mathbb{R}^2 \to \mathbb{R}, \quad f(x,y) = e^{x+y} \]

\[ f(x,y) = \sin(x^2+y^2) \ldots \]

Their graphs are surfaces in \(\mathbb{R}^3\).