Lecture 13 (2/4/2019)

\[ f : \mathbb{R}^n \to \mathbb{R}^n \]

Vector \( v \neq 0 \) is eigenvector of \( f \) if \( f(v) \) is parallel to \( v \), i.e., 
\[ f(v) = \lambda v \text{ for some scalar } \lambda. \]

**Note:** \( f(0) = 0 \)

\[ \text{Why?} \quad f(0) = f(0+0) = f(0) + f(0) \Rightarrow f(0) = 0. \]

**Ex:** If \( f(0) = 0 \) and \( v \neq 0 \) then \( v \) is an eigenvector of \( f \).

\[ \text{Why?} \quad f(v) = \lambda v \]

\[ \text{eigenvalue} \]

**Ex:**
\[ I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \]

\[ I_2v = v \quad \text{for all } v \in \mathbb{R}^2 \]

Every non-zero vector in \( \mathbb{R}^2 \) is an eigenvalue of \( I_2 \).

The corresponding eigenvalue is \( \lambda = 1 \).

**Ex:** \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) is the projection onto the line \( y = 2x \).

There are 2 directions that are preserved under \( f \):
\[ v_1 = (1, 2) \text{ --- parallel to the line} \]
\[ v_2 = (-2, 1) \text{ --- perpendicular to the line} \]

\[ f(v_1) = \lambda v_1 \]

\[ \text{eigenvalue} = 1 \]
\[ f(v_2) = 0 = \lambda v_2 \]

\[ \text{eigenvalue} = 0 \]

In this case, \( f \) has two linearly independent eigenvectors.
Ex: \( f: \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) is the rotation by 20° counterclockwise.

No directions are preserved under \( f \).

\( f \) has no real eigenvectors. But it has two complex eigenvectors.

Ex: \( f: \mathbb{R}^2 \rightarrow \mathbb{R}^2 \), \( f(x, y) = (x+y, y) \)

Only the vectors on the \( x \)-axis has direction preserved under \( f \). The only eigenvector is \( v = (1, 0) \) (and scalar multiples of \( v \)).

\( f(v) = \lambda v \)

**Eigenvalue is** \( \lambda = 1 \)

**How to compute eigenvectors/eigenvalues of a linear map/matrix?**

\[ Av = \lambda v \quad \Rightarrow \quad (A - \lambda \mathbf{I}_n) v = 0 \]

\( n \times n \) non-zero matrix, \( n \times 1 \) column vector

This equation has two solutions: \( v \) and \( 0 \).

The coefficient matrix \( A - \lambda \mathbf{I}_n \) must fail to be invertible.

\[ \text{det} (A - \lambda \mathbf{I}_n) = 0 \]

**Procedure:**

1. Write matrix \( A - \lambda \mathbf{I}_n \).
2. Compute \( \text{det} (A - \lambda \mathbf{I}_n) \). This should be a polynomial of degree \( n \).
3. Find the roots of this polynomial. These are the eigenvalues of \( A \).
4. To each eigenvalue \( \lambda \), find the corresponding eigenvector by solving the equation \( (A - \lambda \mathbf{I}_n) v = 0 \). This equation should have infinitely many solutions.
Example:

\[ A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \]

\[ A - \lambda I_2 = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{bmatrix} \]

\[ \det(A - \lambda I_2) = (1-\lambda)^2 - 4 = (1-\lambda)(3-\lambda) \]

Two roots are \( \lambda_1 = -1 \) and \( \lambda_2 = 3 \). These are the two eigenvalues of \( A \).

Find corresponding eigenvectors:

- For \( \lambda = -1 \):

  We will solve for \( \mathbf{v} \) from the equation \((A - (-1)I_2)\mathbf{v} = 0\).

  Augmented matrix:

  \[ \begin{bmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \end{bmatrix} \]

  \[ R_2 \rightarrow R_2 - R_1 \]

  \[ \begin{bmatrix} 2 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \] (REF)

  This column has no pivot entries

  \( x_2 = t \) (free variable)

  \( x_1 = -t \) (from the first row)

  The eigenvectors corresponding to eigenvalue \( \lambda = -1 \) are \((t, t) = t(-1, 1)\).

- For \( \lambda = 3 \):

  Do similarly.

Example:

\[ A = \begin{bmatrix} 1 & -1 \\ 5 & -3 \end{bmatrix} \]

\[ \det(A - \lambda I_2) = \begin{vmatrix} 1-\lambda & -1 \\ -3 & -3-\lambda \end{vmatrix} = \lambda^2 + 2\lambda + 2 \]

Two complex roots:

\( \lambda_1 = -1 + i \)

\( \lambda_2 = -1 - i \)

Continue next time.