Recall: \( f(x) \approx T_n(x) = f(a) + \frac{f'(a)}{1!} (x-a) + \frac{f''(a)}{2!} (x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!} (x-a)^n \)

\[
= f(a) + \sum_{k=1}^{n} \frac{f^{(k)}(a)}{k!} (x-a)^k
\]

\( n \)th order Taylor poly. of \( f \) about \( a \)

\[
\begin{align*}
\ln(x) &> 0 \\
\frac{d}{dx} \ln(x) &= \frac{1}{x} > x^{-1} \\
\frac{d^2}{dx^2} \ln(x) &= -\frac{1}{x^2}

\frac{d^3}{dx^3} \ln(x) &= -\frac{2}{x^3}

\frac{d^4}{dx^4} \ln(x) &= -\frac{6}{x^4}

\vdots

\frac{d^k}{dx^k} \ln(x) &= -\frac{k!}{x^k}

\end{align*}
\]

\[
\begin{align*}
T_1(x) &= f'(1) (x-1) \\
T_2(x) &= f'(1) (x-1) + \frac{f''(1)}{2!} (x-1)^2 = (x-1) - \frac{1}{2} (x-1)^2
\end{align*}
\]

Taylor polynomials about base point \( a = 1 \):

\( T_0(x) = f(1) = 0 \)

\( T_1(x) = 0 + \frac{f'(1)}{1!} (x-1) = (x-1) \)

1st correction

\( T_2(x) = (x-1) + \frac{f''(1)}{2!} (x-1)^2 = (x-1) - \frac{1}{2} (x-1)^2 \)

2nd correction

\( T_3(x) = (x-1) + \frac{f'''(1)}{3!} (x-1)^3 \)

\( T_4(x) = (x-1) + \frac{f^{(4)}(1)}{4!} (x-1)^4 \)

\( \vdots \)
\[ T_n(x) = \sum_{k=0}^{n} \frac{(-1)^{k+1}}{(k+1)!} (x-a)^{k+1} \]

How to control the error \( R_n(x) = f(x) - T_n(x) \)?

Theorem: (by Lagrange) \( \text{Continuing a} \)

If \( f \) is differentiable \((n+1)\) times on an interval \((a, \beta)\) then

\[ R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} (x-a)^{n+1} \quad (x \in (a, \beta)) \]

for some \( c \) lying between \( a \) and \( x \), depending on \( n \) and \( x \).

This is a generalization of the mean-value theorem (for higher-order derivatives). Why?

Consider \( n = 0 \) (the simplest case):

\[ f(x) - T_0(x) = f(x) - f(a) = f'(c)(x-a) \]

Observation: What does it take to make the error term \( R_n(x) \) small?

Suppose the \((n+1)\)st derivative of \( f \) is bounded, i.e.

\[ |f^{(n+1)}(x)| \leq M \quad \text{for all} \ x \in (a, \beta) \]

then \( R_n(x) \) is small if \( x \) is close to \( a \) or \( n \) is large.
How to estimate the error term of the logarithm function?

\[ f(x) = \ln x, \quad x \in [0.8, 1.2] \]

\[ R_n(x) = f(x) - T_n(x) = \frac{\int_{\theta}^{x} \frac{C}{(x-1)^{n+1}}} \]

\[ = \frac{(1)^{n} C^{-n-1}}{n+1} (x-1)^{n+1} \]

\[ c \in [0.8, 1.2] \]

\[ |R_n(x)| = \frac{|C|^{-(n+1)}}{n+1} (x-1)^{n+1} \leq \frac{(\frac{5}{4})^{n+1}}{n+1} \left( \frac{1}{5} \right)^{n+1} = \frac{1}{n+1} \left( \frac{5}{4} \right)^{n+1} \]

What is the minimum value of \( n \) such that the error doesn't exceed \( 10^{-4} \) for any \( x \in [0.8, 1.2] \)?

\[ \frac{1}{n+1} \left( \frac{1}{4} \right)^{n+1} < 10^{-4} \]

doesn't exceed \( \left( \frac{1}{4} \right)^{n+1} = 4^{-(n+1)} \)

Any value of \( n \) such that \( 4^{-(n+1)} < 10^{-4} \) would do it.

Want: \[ \ln 4^{-(n+1)} < \ln 10^{-3} \]

\[-(n+1) \ln 4 < -4 \ln 10 \]

Divide both sides by \( -\ln 4 \):

\[ n+1 < \frac{4 \ln 10}{\ln 4} \approx 6.64 \]

\( n = 6 \) would do it.