\[ f(x) = \frac{T_n(x)}{n!} + \frac{R_n(x)}{n!} \]

\(n\)th Taylor error term

**Lagrange's theorem:**

\[ R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} (x-x_0)^{n+1} \]

for some \( c \) between \( x \) and \( x_0 \).

Recall:

\[
\frac{\ln 2}{\ln 3} \approx 0 + \frac{x-1}{1} - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \ldots + \frac{(-1)^{n-1}(x-1)^n}{n}
\]

\[ T_n(x) \]

When \( x = 3 \), \( T_n(3) \) is not a good approximation for \( \ln 3 \)
because the \((n+1)\)st correction term is

\[
\frac{(-1)^n \frac{3^n - 1^{n+1}}{n+1}}{n+1} = (-1)^n \frac{2^{n+1}}{n+1}
\]

doesn't tend to zero as \( n \to \infty \).

Assume: \( |f^{(n+1)}(x)| \leq M \) for all \( x \in (x_0-a, x_0+a) \) and for all \( n \).

\[ R_n(x) = f(x) - T_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} \frac{(x-x_0)^{n+1}}{n!} \]

\[ |R_n(x)| \leq \frac{M}{(n+1)!} \alpha^{n+1} \]

For fixed \( n \), \( R_n(x) \to 0 \) as \( x \to x_0 \).

Approximation gets better when \( x \) is close to \( x_0 \).
For fixed $x$, $R_n(x) \to 0$ as $n \to \infty$.

Why? The quotient

\[
\frac{M}{(n+1)!} \frac{x^{n+1}}{n!} = \frac{x}{n+1} \leq \frac{1}{2} \quad \text{for } n \text{ large}
\]

$f(x)$ can be approximated using $f(x_0), f'(x_0), f''(x_0), \ldots, f^{(n)}(x_0)$. When $x$ is further from $x_0$, $n$ is required to be bigger in order to keep the error term small.

* Observation: $f(x) = T_n(x) + R_n(x) = \lim_{n \to \infty} T_n(x)$

Formally,

\[
f(x) = f(x_0) + \frac{f'(x_0)}{1!} (x-x_0) + \frac{f''(x_0)}{2!} (x-x_0)^2 + \frac{f'''(x_0)}{3!} (x-x_0)^3 + \ldots
\]

an infinite sum, called Taylor Series

How to define an infinite sum?

The sum $a_1 + a_2 + a_3 + \ldots$ is called an infinite sum, or series.

The sum $S_n = a_1 + a_2 + \ldots + a_n$ is called a partial sum.

Def:

If $\lim S_n = S$ exists then the series $\sum_{n=1}^{\infty} a_n$ is said to converge and $\sum_{n=1}^{\infty} a_n = S$ is the sum of the series.

If $\lim S_n = \pm \infty$ or doesn't exist then the series is said to diverge.

Ex:

Consider the infinite sum $1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^5} + \ldots$
Partial sum: \( S_n = 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n} \)

How to compute \( S_n \)?

\[
2S_n = 2 + 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}}
\]

\[
S_n = \quad 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}} + \frac{1}{2^n}
\]

\[
S_n = 2 - \frac{1}{2^n}
\]

Because \( \lim S_n = 2 \), the series converges to 2.

\[
1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \cdots = 2
\]

**Ex.** Consider the series \( 1 + q + q^2 + q^3 + q^4 + \cdots \).

Partial sum \( S_n = 1 + q + q^2 + \cdots + q^n \)

How to compute \( S_n \)?

\[
qS_n = \quad q + q^2 + q^3 + \cdots + q^n + q^{n+1}
\]

\[
S_n = 1 + q + q^2 + \cdots + q^n \]

\[
(q-1)S_n = q^{n+1} - 1
\]

\[
S_n = \frac{q^{n+1} - 1}{q-1}
\]

This sequence converges to \( \frac{-1}{q-1} \) if \(-1 < q < 1\).

It doesn’t converge if \( q \leq -1 \) or \( q > 1 \).

\[
1 + q + q^2 + q^3 + \cdots = \frac{1}{1-q} \quad \text{for all } q \in (-1, 1).
\]