Question: Is it true that \( \sum_{n=1}^{\infty} a_n b_n = \left( \sum_{n=1}^{\infty} a_n \right) \left( \sum_{n=1}^{\infty} b_n \right) \)?

No, take \( a_n = 1 \) (for all \( n \)) and \( b_n = \frac{1}{2^n} \). Even finite sum doesn't have this property, for example with \( n=2 \):

\[
a_1 b_1 + a_2 b_2 = (a_1 + a_2)(b_1 + b_2)
\]

Convergence tests: quick ways to check if a series converges or diverges.

"General term goes to zero" test:

- If \( \sum a_k \) converges, then \( a_k \to 0 \) as \( n \to \infty \).

Why? \( S_n = a_1 + a_2 + \ldots + a_n \)

the series converges \( \Rightarrow \) \( S_n \) converges

this implies that the increment of the sum goes to \( 0 \).

\[
S_{n+1} - S_n = a_{n+1} : \text{increment at } n^{th} \text{ step}
\]

Therefore, \( a_n \) must go to \( 0 \).

- If \( a_n \not\to 0 \) as \( n \to \infty \) then \( \sum a_k \) doesn't converge.

Eq.:

\[
\sum_{n=1}^{\infty} \frac{n^2}{n^2+1}
\]

The general term of the series is \( a_n = \frac{n^2}{n^2+1} \). This term is an increment of the sum at the \((n-1)^{th}\) step.

\[
a_n \to 1 \neq 0 \text{ as } n \to \infty
\]

Thus, the series diverges.
Ex: \[ \sum_{n=1}^{\infty} \frac{(-1)^n n^2}{n^2+1} \] (alternating series)

General term \[ a_n = \frac{(-1)^n n^2}{n^2+1} \] doesn't converge to 0. Why?

\[ |a_n| = \frac{n^2}{n^2+1} \rightarrow 1 \neq 0 \]

The series diverges.

*Warning:* If \( a_n \rightarrow 0 \) as \( n \rightarrow \infty \), it's *not* necessarily true that \( \sum a_n \) converges.

For example, the series \( \sum \frac{1}{n^2} \) diverges because the partial sum

\[ s_n = \frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{2n} > 2 \left( \frac{1}{2n} - 1 \right) \]

diverges (Homework 5).

**Comparison Test:**

- If \( |a_n| \leq b_n \) for all \( n \) sufficiently large \( (n \geq N) \)

and \( \sum b_n \) converges then \( \sum a_n \) also converges.

(A series dominated by a convergent series is convergent)

- If \( 0 \leq a_n \leq b_n \) for all \( n \) sufficiently large \( (n \geq N) \)

and \( \sum a_n \) diverges then \( \sum b_n \) also diverges.

(A series that dominates a divergent series with nonnegative terms is also divergent)
\[ \sum_{n=0}^{\infty} \frac{(-1)^n}{n^2+1} \]

\[ |a_n| = \frac{1}{n^2+1} \leq \frac{1}{n^2} = \frac{1}{n^2} \]

The series \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) converges (shown in last lecture).

Therefore, the series \( \sum_{n=0}^{\infty} \frac{(-1)^n}{n^2+1} \) converges.

\[ \sum_{n=3}^{\infty} \frac{1}{2^n - n^2} \]

We notice that as \( n \) large, \( n^2 \) is dominated by \( 2^n \), so the fraction \( \frac{1}{2^n - n^2} \) is essentially \( \frac{1}{2^n} \) as \( n \) large. Thus, we guess that the given series converges. How to prove?

Guess:\[ \frac{1}{2^n - n^2} < \frac{2}{2^n} \]

This is equivalent to \( 2^n < 2(2^n - n^2) \), which is equivalent to \( 2n^2 < 2^n \), which is a true inequality for \( n \) sufficiently large.

Thus, \( \sum_{n=3}^{\infty} \frac{1}{2^n - n^2} \) is dominated by \( \sum_{n=3}^{\infty} \frac{2}{2^n} \), which is a convergent series.