System of linear equations:
\[
\begin{align*}
2x + y &= -2 \\
3x + 2y &= 4
\end{align*}
\]

Matrix form:
\[
\begin{bmatrix}
2 & 1 \\
3 & 2
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix}
= 
\begin{bmatrix}
-2 \\
4
\end{bmatrix}
\]

Augmented matrix:
\[
[A|b] = \begin{bmatrix}
2 & 1 & -2 \\
3 & 2 & 4
\end{bmatrix}
\]

This encodes all information needed to solve the system.

We want to eliminate \(x\) from the second equation. This is done by several ways: one can subtract \(\frac{3}{2}\) times the first row from the second row, which gives
\[
\begin{bmatrix}
2 & 1 & -2 \\
0 & \frac{1}{2} & 7
\end{bmatrix}
\]

Or one can divide the first row by 2, then subtract 3 times the first row from the second row. This method gives
\[
\begin{bmatrix}
1 & \frac{1}{2} & -1 \\
0 & \frac{1}{2} & 7
\end{bmatrix}
\]

Both systems are equivalent to the original one, and equivalent to each other.

Then we do back substitution:
\[
\begin{bmatrix}
2 & 1 & -2 \\
0 & \frac{1}{2} & 7
\end{bmatrix}
\]

The second eq. gives \(\frac{1}{2}y = 7\). Thus \(y = 14\).

The first eq. gives \(2x + y = -2\). Thus, \(x = \frac{-2 - y}{2} = -7\).
Elementary row operations:

1) Multiply a row by a nonzero number
2) Interchange two rows.
3) Replace a row by the sum of itself and a multiple of another row.

Any of these operations make sure that the new system is equivalent to the old one.

Strategy:

- Write the augmented matrix of the system.
- Use row operations consecutively to bring the original augmented matrix into row echelon form:

\[
\begin{bmatrix}
& * \\
& all \\
& zero here
\end{bmatrix}
\]

- Back substitution from bottom to top.

Definition:

The first nonzero entry of a row (counting from the left) is called a pivot entry.

A matrix is in row echelon form if:

- for any two consecutive rows, the pivot entry of the row above is further to the left of the pivot entry of the row below.
- all zero rows are gathered at the bottom.

Examples provided on another sheet (check course website).
For:

\[
\begin{align*}
x_1 + 2x_2 + 3x_3 &= 9 \\
2x_1 - x_2 + x_3 &= 9 \\
3x_1 - x_2 &= 3
\end{align*}
\]

Augmented matrix of the system:

\[
\begin{bmatrix}
1 & 2 & 3 & | & 9 \\
2 & -1 & 1 & | & 8 \\
3 & 0 & -1 & & 3
\end{bmatrix}
\]

\[
\begin{align*}
R_2 &= R_2 - 2R_1 \\
R_3 &= R_3 - 3R_1
\end{align*}
\]

\[
\begin{bmatrix}
1 & 2 & 3 & | & 9 \\
0 & -5 & -5 & | & -10 \\
0 & -6 & -10 & & -24
\end{bmatrix}
\]

\[
\begin{align*}
R_2 &= \frac{R_2}{5} \\
R_3 &= R_3 + 6R_2
\end{align*}
\]

\[
\begin{bmatrix}
1 & 2 & 3 & | & 9 \\
0 & 1 & 1 & | & 2 \\
0 & 0 & -4 & & -12
\end{bmatrix}
\]

Row echelon form.

Then do back substitution:

3rd row: \(-4x_3 = -12 \Rightarrow x_3 = 3\)

2nd row: \(x_1 + x_2 + 2x_3 = 9 \Rightarrow x_1 = 9 - 2x_2 - 3x_3 = 9 - 2(-1) - 3(3) = 2\)

1st row: \(x_1 - x_2 + x_3 = 1 \Rightarrow x_1 = 1 - x_2 - x_3 = 1 - (-2) - 3 = 0\)

For:

\[
\begin{align*}
x_1 - x_2 + x_3 &= 1 \\
2x_1 + x_2 &= 2 \\
4x_1 - x_2 + 2x_3 &= 3
\end{align*}
\]

Augmented matrix:

\[
\begin{bmatrix}
1 & -1 & 1 & | & 1 \\
2 & 1 & 0 & | & 2 \\
4 & -1 & 2 & | & 3
\end{bmatrix}
\]

\[
R_2 = R_2 - 2R_1
\]

\[
\begin{bmatrix}
1 & -1 & 1 & | & 1 \\
0 & 3 & -2 & | & 0 \\
4 & -1 & 2 & | & 3
\end{bmatrix}
\]

\[
R_3 = R_3 - 4R_1
\]

\[
\begin{bmatrix}
1 & -1 & 1 & | & 1 \\
0 & 3 & -2 & | & 0 \\
0 & 8 & -2 & | & -1
\end{bmatrix}
\]
\[ R_3 = R_2 - R_2 \rightarrow \begin{bmatrix} 1 & -1 & -1 & 1 \\ 0 & 3 & -2 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \text{ row echelon form} \]

Back substitution: the last row implies \( 0 = -1 \) (!)

The system is inconsistent (having no solutions).

The system is inconsistent if there appears a row \([ 0 \ 0 \ \ldots \ 0 \ 1 \ a \ ]\).

3 planes have no common points.