Lecture 7 (1/23/2019)

* System with infinitely many solutions:

Ex:

\[
\begin{align*}
2x_1 - x_2 + x_3 &= 0 \\
2x_1 + x_2 - x_3 &= 1 \\
x_1 - x_2 + x_3 &= 1 \\
\end{align*}
\]

Augmented matrix:

\[
\begin{bmatrix}
1 & -1 & 1 & | & 0 \\
2 & 1 & -1 & | & 1 \\
4 & -1 & 1 & | & 1 \\
\end{bmatrix}
\]

\[
\begin{align*}
&\rightarrow R_2 = R_2 - 2R_1 \\
&\rightarrow R_3 = R_3 - 4R_1 \\
&\rightarrow R_3 = R_3 - R_2 \\
\end{align*}
\]

\[
\begin{bmatrix}
1 & -1 & 1 & | & 0 \\
0 & 3 & -3 & | & 1 \\
0 & 0 & 0 & | & 0 \\
\end{bmatrix}
\]

row echelon form

The columns without pivot entries give us a free variable. In this case, \( x_3 \) is a free variable and there are infinitely many solutions.

Back substitution:

3rd row: no useful information

2nd row: \( 3x_2 - 3x_1 = 1 \Rightarrow x_2 = \frac{1 + 3t}{3} = \frac{1}{3} + t \)

1st row: \( x_1 - x_2 + x_3 = 0 \Rightarrow x_1 = x_2 - x_3 = \frac{1}{3} \)

Thus,

\[
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
\end{bmatrix} = \begin{bmatrix}
\frac{1}{3} \\
\frac{1}{3} + t \\
t \\
\end{bmatrix} = \begin{bmatrix}
\frac{1}{3} \\
\frac{1}{3} \\
0 \\
\end{bmatrix} + t \begin{bmatrix}
0 \\
1 \\
1 \\
\end{bmatrix}
\]

parametric vector form
Ex: A system of 3 equations and 4 unknowns has augmented matrix \([A\mid b]\). Suppose that after several row reduction steps, one gets:

\[
\begin{bmatrix}
1 & 2 & 3 & 4 & 0 \\
0 & 2 & -1 & 0 & 2 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

How many solutions are there?

Answer: no solutions. Although the 3rd and 4th cols are without pivot entries, the system is inconsistent due to the last row.

Ex: 

\[
\begin{bmatrix}
1 & 2 & 3 & 4 & 0 \\
0 & 2 & -1 & 0 & 2 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

How many solutions are there?

Answer: infinitely many.

The 3rd and 4th cols are without pivot entries. Thus, \(x_3\) and \(x_4\) are free variables.

\[x_3 = s, \quad x_4 = t\]

Back substitution:

3rd row: no useful information

2nd row: \(2x_2 - x_3 = 2 \Rightarrow x_2 = \frac{2 + x_3}{2} = 1 + \frac{5}{2}x_3\).

1st row: \(x_1 + 2x_2 + 3x_3 + 4x_4 = 0 \Rightarrow x_1 = -2s - 3t - 4t = -2 - 4s - 4t\)

Conclusion:
\[
\begin{bmatrix}
-2 & -4 & -4 \\
1 & 5 \\
-1 & -4 \\
0 & 0
\end{bmatrix}
= \begin{bmatrix}
-2 \\
1 \\
-1 \\
0
\end{bmatrix} + \begin{bmatrix}
-4 \\
5 \\
-4 \\
0
\end{bmatrix}
= \begin{bmatrix}
-2 \\
1 \\
-1 \\
0
\end{bmatrix} + t \begin{bmatrix}
-4 \\
5 \\
-4 \\
0
\end{bmatrix}
\]

parametric vector form

System of linear equations

no solutions \[\text{inconsistent}\]

1 sol. \[\text{consistent}\]

infinitely many \[\text{consistent}\]

These are all possible scenarios.

no sols: when there occurs a row \( \begin{bmatrix} 0 & 0 & \ldots & 0 & | & c \end{bmatrix} \)

inf. many sols: when there are no such rows and there is at least one col. without pivot entries.

* Reduced row echelon form of a matrix (RREF):*

RREF is REF (row echelon form) with two additional properties:

* All pivot entries must be 1 (called pivot 1)

* If a column contains a pivot 1, all other entries on that column must be 0.

A column containing a pivot 1 is said to be a pivot column.

**E2:** see the example sheet posted on course website.