Homework 1
Due 01/17/2020

In the following problems, make sure to write your arguments coherently in full sentences. Start a sentence with words rather than a formula. Use words to transition your ideas, for example “This leads to”, “Therefore”, “We want to show”, etc.

1. Let $V$ be a vector space over a field $F$ (which is $\mathbb{Q}$, $\mathbb{R}$, or $\mathbb{C}$). Use the axioms of vector space to show the following properties. Make sure to mention which axiom(s) you use.
   
   (a) (Cancellation law) If $u_1, u_2, v \in V$ and $u_1 + v = u_2 + v$, then $u_1 = u_2$.
   
   (b) (Uniqueness of zero element) If $a$ and $b$ are neutral elements of $V$, i.e.
   
   \[ a + v = v \quad \forall v \in V, \]
   \[ b + v = v \quad \forall v \in V, \]
   
   then $a = b$.
   
   *Note: Because the neutral element is unique, it is denoted by 0 and is called the zero vector.*

   (c) (Scaling by 0)
   
   \[ 0v = 0 \quad \forall v \in V. \]

   (d) (Additive inverse) If $v, w \in V$ satisfy $v + w = 0$ then $w = (-1)v$ (vector $v$ scaled by factor $-1$).
   
   *Note: the additive inverse of $v$ is denoted as $-v$."

2. On the set of complex numbers $\mathbb{C}$, we define another product rule as follows:

   \[ z \ast v = \bar{z}v \quad \forall z, v \in \mathbb{C}. \]

   The star denotes the new product rule. The product on the right hand side is the usual product of complex numbers. Here $\bar{z}$ denotes the complex conjugate of $z$. Show that $V = \mathbb{C}$ is a vector space over $F = \mathbb{C}$ under the usual addition and the new product rule.

3. Let $F$ be a field of numbers. Put

   \[ V = \{ A \in M_{2 \times 2}(F) : A + A^T = 0 \}. \]

   (a) Show that $V$ is a vector space over $F$. Here $A^T$ denotes the transpose of matrix $A$.
   
   (b) Find a basis and the dimension of $V$.

   Do the following problem for 6 bonus points.

4. Let $V = \mathbb{Q}^{(1,3) \cap \mathbb{Q}}$, which is the set of all functions from $(1,3) \cap \mathbb{Q}$ to $\mathbb{Q}$. Recall that $V$ is a vector space over $F = \mathbb{Q}$.

   (a) Do the functions $f(x) = \frac{x}{x-2}$ and $g(x) = \sqrt{x}$ belong to $V$?
   
   (b) Consider three functions $f_1(x) = x - 1$, $f_2(x) = x$, and $f_3(x) = 1/x$. They are vectors in $V$. Show that $f_1$, $f_2$, $f_3$ are linearly independent.