Homework 2
Due 01/27/2020

In the following problems, make sure to write your arguments coherently in full sentences. Start a sentence with words rather than a formula. Use words to transition your ideas, for example “This leads to”, “Therefore”, “We want to show”, etc.

1. Let $U$ and $V$ be subspaces of a vector space $W$.
   (a) The sum of $U$ and $V$, denoted by $U + V$, is defined as the set $U + V = \{u + v : u \in U, v \in V\}$. Show that $U + V$ is a subspace of $W$.
   (b) Let $w$ be a vector in $W$ but not in $V$. Show that $w + v \notin V \quad \forall v \in V$.
       Hint: proof by contradiction. (Suppose that the conclusion is false. Then use valid arguments to find a contradiction.)
   (c) Show that the union $U \cup V$ is a subspace of $W$ if and only if either $U \subset V$ or $V \subset U$.
       Hint: proof by contradiction.
   (d) The sum $U + V$ is said to be a direct sum if each vector in $U + V$ can be written in only one way as $u + v$ where $u \in U$ and $v \in V$. (“Only one way” means that if $x \in U + V$ and $x = u_1 + v_1 = u_2 + v_2$ for some $u_1, u_2 \in U$ and $v_1, v_2 \in V$ then $u_1 = u_2$ and $v_1 = v_2$.)
       Show that $U + V$ is a direct sum if and only if $U \cap V = \{0\}$.

2. Let $F$ be a field of numbers ($\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$). Show that the polynomials $1, 1 + x, 1 + x + x^2, 1 + x + x^2 + x^3$ form a basis of $P_3(F)$ (the space of all polynomials of degree $\leq 3$ with coefficients in $F$).

3. Let $U$ and $V$ be vector spaces over a field $F$. Let $f : U \rightarrow V$ be a linear map.
   (a) Let $u_1, u_2, \ldots, u_k \in U$. Show that if $f(u_1), f(u_2), \ldots, f(u_k)$ are linearly independent then $u_1, u_2, \ldots, u_k$ are also linearly independent.
   (b) $f$ is said to be injective, or monomorphic, if $f(u) = f(v)$ implies $u = v$. Show that $f$ is monomorphic if and only if null$(f) = \{0\}$.
   (c) Suppose $f$ is monomorphic. Show that $f$ preserves linear independence. That is to show: if $u_1, u_2, \ldots, u_k \in U$ are linearly independent then $f(u_1), f(u_2), \ldots, f(u_k)$ are also linearly independent.

Do the following problem for 6 bonus points.

4. Let $V$ be a vector space over $F$ and $f : V \rightarrow F$ be a linear map. Let $v$ be a vector in $V$ but not in null$(f)$. Show that $V = \text{null}(f) + \text{span}\{v\}$. Is this sum a direct sum?