Homework 3
Due 02/03/2020

In the following problems, make sure to write your arguments coherently in full sentences. Start a sentence with words rather than a formula. Use words to transition your ideas, for example “This leads to”, “Therefore”, “We want to show”, etc.

1. Consider a map \( G : P_2(\mathbb{R}) \to P_2(\mathbb{R}) \) given by \( G(u) = (x+1)u' - 2u \).
   (a) Show that \( G \) is a linear map.
   (b) Find a basis and the dimension of null(\( G \)). What is the nullity of \( G \)?
   (c) Find a basis and the dimension of range(\( G \)). What is the rank of \( G \)?
   (d) Is \( G \) a monomorphism, epimorphism, isomorphism or none of them?

2. Let \( V = \{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2\times 2}(\mathbb{C}) : a + b + c + id = 0 \} \).
   Consider a linear map \( H : V \to P_2(\mathbb{C}) \) given by
   \[
   H \left( \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \right) = (a + b)z^2 + (b + c)z + (c + d).
   \]
   (a) Show that \( V \) is a subspace of \( M_{2\times 2}(\mathbb{C}) \).
   (b) Find a basis of \( V \).
   (c) Find a matrix representation of \( H \).
   (d) Find the nullity of \( H \).
   (e) Find the rank of \( H \).
   Hint: use the rank-nullity theorem

3. Let \( V \) be the subspace of \( M_{2\times 2}(\mathbb{R}) \) consisting of all matrices in which the sum of entries on each row is equal to 0. Let \( W \) be the subspace of \( M_{2\times 2}(\mathbb{R}) \) consisting of all matrices in which the sum of entries on each column is equal to 0. Find a basis of \( V + W \).

Do the following problem for 6 bonus points.

4. Let \( V \) be a vector space with basis \( B_1 = \{ v_1, v_2, \ldots, v_7 \} \), and \( W \) be a vector space with basis \( B_2 = \{ w_1, w_2, \ldots, w_6 \} \). Let \( f : V \to W \) be a linear map given by

\[
\begin{align*}
 f(v_1) &= w_1 + w_2 - w_4 + 2w_6, \\
f(v_2) &= 3w_1 - w_2 - w_3 + w_5 - 4w_6, \\
f(v_3) &= 2w_2 + 5w_3 - w_4 + 7w_5 - w_6, \\
f(v_4) &= w_1 + w_3 - w_4 + w_6, \\
f(v_5) &= w_2 - 4w_4 + 5w_5 + 3w_6, \\
f(v_6) &= w_1 + w_2 + 2w_3 + 3w_4 + 5w_5, \\
f(v_7) &= 2w_1 - 6w_3 + 2w_4 + w_5 - w_6
\end{align*}
\]

(a) Write the matrix that represents \( f \) relative to bases \( B_1 \) and \( B_2 \).
(b) Find the rank and nullity of \( f \). (You are encouraged to use Matlab to do this problem. If you use Matlab, please write down the Matlab commands and the outputs.)