Homework 6
Due 02/28/2020

In the following problems, make sure to write your arguments coherently in full sentences. Start a sentence with words rather than a formula. Use words to transition your ideas, for example “This leads to”, “Therefore”, “We want to show”, etc.

1. Let \( V \) be an inner product space and let \( \| \cdot \| \) be the norm associated with the inner product on \( V \), i.e. given by \( \|v\| = \sqrt{(v,v)} \). Show that
   (a) (Parallelogram identity)
   \[
   \|u + v\|^2 + \|u - v\|^2 = 2(\|u\|^2 + \|v\|^2) \quad \forall u, v \in V.
   \]
   (b) (Cauchy-Schwarz inequality)
   \[
   |(u,v)| \leq \|u\|\|v\| \quad \forall u, v \in V.
   \]
   Note: for Part (b), you only need to show proof for the case \( F = \mathbb{R} \) or \( \mathbb{Q} \). Use the fact that \( (tu + v, tu + v) \geq 0 \) for all \( t \in \mathbb{R} \). If you can do the case \( F = \mathbb{C} \), you will be granted 3 extra points.

2. On \( \mathbb{R}^n \), let us consider an operator \( \| \cdot \| \) given by \( \|x\| = |x_1| + |x_2| + \ldots + |x_n| \) where
   \[
   x = \begin{bmatrix}
   x_1 \\
   \vdots \\
   x_n
   \end{bmatrix}
   \]
   Show that \( \| \cdot \| \) is a norm on \( \mathbb{R}^n \). (This is known as a “taxicab” norm.)

3. Let \( V \) be an inner product space. For each subset \( S \subset V \), the orthogonal complement of \( S \) is denoted by \( S^\perp = \{ v \in V : v \perp w \text{ for all } w \in S \} \). Let \( U \) be a subspace of \( V \). Show the following statements:
   (a) \( U^\perp \) is a subspace of \( V \).
   (b) \( U \oplus U^\perp = V \).
   (c) \( (U^\perp)^\perp = U \).

4. Let \( V \) be an inner product space over \( \mathbb{C} \) with an orthonormal basis \( \mathcal{B} = \{ v_1, v_2, \ldots, v_n \} \). Let \( x = \sum_{k=1}^{n} \alpha_k v_k \) and \( y = \sum_{k=1}^{n} \beta_k v_k \) where \( \alpha_k, \beta_k \in \mathbb{C} \) for \( k = 1, 2, \ldots, n \). Show that
   \[
   (x,y) = \sum_{k=1}^{n} \alpha_k \overline{\beta_k}.
   \]

Do the following problem for 6 bonus points.

5. On \( P_n(\mathbb{R}) \), which is the space of polynomials of real coefficients with degree \( \leq n \), let us consider the inner product
   \[
   (f,g) = \int_{0}^{1} f(x)g(x)dx.
   \]
   Let \( p \) be the orthogonal projection of \( x^2 \) on the line span\{\( x \)\}. Determine \( \|x^2 - p\| \).