Last time, we defined invariant subspace of a vector space with respect to a linear map. Consider a linear map \( f : V \rightarrow V \). A subspace \( W \) of \( V \) is said to be invariant under \( f \) if \( f(W) \subseteq W \).

A special invariant subspace of \( V \) is known as the eigenspace. Let \( \lambda \in \mathbb{F} \), which is the base field of \( V \). The set

\[
E_\lambda = \{ v \in V : f(v) = \lambda v \}
\]

consists of all vectors in \( V \) that get scaled by a factor of \( \lambda \) when applying \( f \).

\( E_\lambda \) is:

- (a) a subspace of \( V \),
- (b) invariant under \( f \).

The proof of (a) follows from the standard procedure: to check if \( E_\lambda \) is a subspace of \( V \), knowing that it is already a subset of \( V \), we only need to check 3 properties:

1) \( 0 \in E_\lambda \)
2) \( E_\lambda \) is closed under addition.
3) \( E_\lambda \) is closed under scaling

How to check 2)? Let \( v, w \in E_\lambda \). We want to show \( v+w \in E_\lambda \).

Because \( v, w \in V \), we have

\[
f(v) = \lambda v, \quad f(w) = \lambda w.
\]

To show that \( v+w \in E_\lambda \), we need to show \( f(v+w) = \lambda(v+w) \).

We have

\[
\text{LTS} = f(v) + f(w) \quad \text{(because } f \text{ is linear)}
= \lambda v + \lambda w \quad \text{(because } v, w \in E_\lambda) \\
= \lambda (v+w) \\
= \text{RHS}.
\]
Thus, we have proved 2) Why is 1) true? Notice that \( f(0) = 0 = \lambda 0 \).
Thus, \( 0 \in E_\lambda \) Property 3) can be proven in the same manner as Property 2).

(b) How to show that \( E_\lambda \) is invariant under \( f \)?
Let \( v \in E_\lambda \). We show that \( f(v) \in E_\lambda \).

But \( w = f(v) \). To show that \( w \in E_\lambda \), we need to show \( f(w) = \lambda w \).
Because \( v \in E_\lambda \), we have \( f(v) = \lambda v \). In other words, \( w = \lambda v \).
We have
\[
    f(w) = f(\lambda v) = \lambda f(v) = \lambda w
\]
which is what we wanted to show. □

For most \( \lambda \), \( E_\lambda \) is equal to \([0,1] \), the trivial vector space. There are some very special values \( \lambda \) such that \( E_\lambda \neq [0,1] \). Those values are called eigenvalues of \( f \). If \( \lambda \) is an eigenvalue of \( f \) then \( E_\lambda \) is called eigenspace corresponding to \( \lambda \). A nonzero element of \( E_\lambda \) is called an eigenvector corresponding to \( \lambda \).

In other words, an eigenvector of \( f \) corresponding to \( \lambda \) is a vector that gets scaled by factor \( \lambda \) when applying \( f \).

An eigenvector of \( f \) is not rotated by \( f \), only scaled. In \( E_\lambda \), every vector is scaled by factor \( \lambda \).
The notion of eigenvalue and eigenvectors of a linear map is a generalization of the notion of eigenvalues and eigenvectors of a matrix, which we already learned in Linear Algebra I. The connection between them is as follows.

If \( A \in \mathbb{M}_{n \times n}(F) \) then the eigenvalues and eigenvectors of \( A \) are exactly the eigenvalues and eigenvectors of the linear map \( f: \mathbb{R}^n \to \mathbb{R}^n \) given by \( f(v) = Av \).

Why so? If \( \lambda \) is an eigenvalue of \( A \) then there is some \( v \neq 0 \) such that \( (A - \lambda I_n)v = 0 \). This is equivalent to \( Av = \lambda v \) or simply \( Av = \lambda v \). Thus, \( f(v) = \lambda v \). This means \( v \in E_\lambda \) and \( v \neq 0 \). Then \( \lambda \) is an eigenvalue of \( f \). One can also show that the converse is true: an eigenvalue of \( f \) is also an eigenvalue of \( A \).

More generally, we have the following:

Let \( f: V \to V \) be a linear map. Let \( B \) be a basis of \( V \), and \( A = [f]_B \) be the matrix that represents \( f \) in basis \( B \). Then the eigenvalues of \( f \) are the same as the eigenvalues of \( A \) and vice versa.

How do we see this? Let \( \lambda \in F \) be an eigenvalue of \( f \). By the definition of eigenvalues, we have \( E_\lambda \neq \{0\} \). This means there is a nonzero vector \( v \in V \) such that

\[
[f(v)]_B = \lambda [v]_B.
\]

Let us take the coordinate of both sides in basis \( B \).

\[
[f(v)]_B = [\lambda v]_B.
\]

We know that

\[
\text{LHS} = [f(v)]_B = [f]_B [v]_B = A [v]_B,
\]

\[
\text{RHS} = [\lambda v]_B = \lambda [v]_B.
\]
Hence, we obtain an equation

$$A \mathbf{v}_g = \lambda \mathbf{v}_g.$$  \((\star)\)

Note that \(\mathbf{v}_g \neq 0\) because \(v \neq 0\). From \((\star)\), we realize that \(\mathbf{v}_g\) is an eigenvector of \(A\) and \(\lambda\) is the corresponding eigenvalue. We have showed that an eigenvalue of \(f\) is an eigenvalue of \(A\). The converse can be shown similarly: an eigenvalue of \(A\) is also an eigenvalue of \(f\).

We have found a connection between the eigenvalues of \(f\) and the eigenvalues of the matrix that represents \(f\): they are the same. How about the eigenspace \(E_1\) of \(f\) and the eigenspace \(E_A\) of \(A = [f]_g\)? What is the relation between them? We know that

\[
E_f = \{ v \in V : f(v) = \lambda v \},
\]
\[
E_A = \{ x \in \mathbb{R}^n : Ax = \lambda x \}. \quad \text{(Here } n = \dim V).\]

From \((\star)\), we have

\[
E_A = \{ \mathbf{v}_g : v \in E_f \}.
\]

In other words, \(E_A\) is the set of all the coordinates (in basis \(B\)) of the eigenvectors of \(f\).

This observation gives us a method to find the eigenvalues and eigenvectors of \(f\) by finding the eigenvalues and eigenvectors of the matrix \(A = [f]_g\). The procedure is as follows.

1. Given a function \(f : V \rightarrow V\), fix a basis \(B\) of \(V\) and find the matrix \([f]_B\). Call it \(A\).
2. Find the eigenvalue \(\lambda\)'s and the corresponding eigenspaces \(E_A\) of \(A\). We know how to do this in Linear Algebra I.
3. For each eigenvalue \(\lambda\), find a basis of \(E_A\).
4. Then convert this basis to a basis of \(E_A\).
Ex: Let \( f: M_{2 \times 2}(\mathbb{R}) \to M_{2 \times 2}(\mathbb{R}) \) be a linear map given by
\[
    f\left(\begin{array}{cc}
        a & b \\
        c & d
    \end{array}\right) = \begin{array}{cc}
        d & -b \\
        -c & a
    \end{array}.
\]
Find the eigenvalues and the corresponding eigenspaces of \( f \).

Fix a standard basis of \( M_{2 \times 2}(\mathbb{R}) \):
\[
    B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}
\]
\( E_1 \) \( E_2 \) \( E_3 \) \( E_4 \)

We have
\[
    f(E_1) = f\left(\begin{array}{cc}
        1 & 0 \\
        0 & 0
    \end{array}\right) = \begin{array}{cc}
        0 & 0 \\
        0 & 0
    \end{array} = E_4
\]
\[
    f(E_2) = f\left(\begin{array}{cc}
        0 & 1 \\
        0 & 0
    \end{array}\right) = \begin{array}{cc}
        0 & -1 \\
        0 & 0
    \end{array} = -E_2
\]
\[
    f(E_3) = f\left(\begin{array}{cc}
        0 & 0 \\
        1 & 0
    \end{array}\right) = \begin{array}{cc}
        0 & 0 \\
        -1 & 0
    \end{array} = -E_3
\]
\[
    f(E_4) = f\left(\begin{array}{cc}
        0 & 0 \\
        0 & 1
    \end{array}\right) = \begin{array}{cc}
        1 & 0 \\
        0 & 0
    \end{array} = E_1
\]

Thus, the matrix that represents \( f \) in basis \( B \) is
\[
    [f]_B = \begin{bmatrix}
        [f(E_1)]_B & [f(E_2)]_B & [f(E_3)]_B & [f(E_4)]_B
    \end{bmatrix} =
    \begin{bmatrix}
        0 & 0 & 0 & 1 \\
        0 & -1 & 0 & 0 \\
        0 & 0 & -1 & 0 \\
        1 & 0 & 0 & 0
    \end{bmatrix}
\]
\( A \)

What are the eigenvalues of \( A \)?
\[ \det(A - \lambda I_4) = \begin{vmatrix} -\lambda & 0 & 0 & 1 \\ 0 & -1 -\lambda & 0 & 0 \\ 0 & 0 & -1 -\lambda & 0 \\ 1 & 0 & 0 & -\lambda \end{vmatrix} \]

\[ = \ldots \]

\[ = (\lambda + 1)^3 (\lambda - 1). \]

One can use Matlab to compute the eigenvalues of \( A \) as follows:

\[ \Rightarrow A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \]

\[ \Rightarrow \text{eig}(A) \]

\( A \) has two eigenvalues: \( \lambda = \pm 1 \).

- Find eigenspace \( \widetilde{E}_1 \) (corresponding to \( \lambda = 1 \)):
  
  To find \( \widetilde{E}_1 \), we find all vectors \( x \in \mathbb{R}^4 \) that satisfy

  \[ Ax = 1 \cdot x. \]

  This equation is equivalent to \( (A - I_4)x = 0 \)

  We are finding the null space of matrix \( A - I_4 \).

  \[ A - I_4 = \begin{bmatrix} -1 & 0 & 0 & 1 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix} \]

  \[ \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \]

  \[ x_1 \ x_2 \ x_3 \ x_4 \]

  We get \( x_1 = x_4 \), \( x_2 = x_3 = 0 \) Thus,

  \[ \widetilde{E}_1 = \left\{ (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 = x_4, x_2 = x_3 = 0 \right\} \]

  \[ = \left\{ (x_1, 0, 0, x_4) : x_1 \in \mathbb{R} \right\} \]

  \[ = \text{span} \left\{ (1, 0, 0, 1) \right\}. \]

  Thus, \( \widetilde{E}_1 \) has basis \( \{ (1, 0, 0, 1) \} \).

  - Find eigenspace \( \widetilde{E}_-1 \) of \( A \):

    We get \( \widetilde{E}_{-1} = \left\{ (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 = -x_4 \right\} \)

    \[ = \left\{ (x_1, x_2, x_3, -x_4) : x_1, x_2, x_3 \in \mathbb{R} \right\} \]

    \[ = \text{span} \left\{ (1, 0, 0, -1), (0, 1, 0, 0), (0, 0, 1, 0) \right\}. \]
One can use Matlab to assist the computation of the eigenvalues and eigenvectors of $A$ as follows:

$$
\Rightarrow A = \begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix}
$$

$$
\Rightarrow [\mathbf{a}, \mathbf{b}] = \text{eig}(A)
$$

Matlab returns:

$$
\mathbf{a} = \begin{bmatrix}
0 & 0 & 0.7071 & 0.7071 \\
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & -0.7071 & 0.7071
\end{bmatrix}
$$

$$
\mathbf{b} = \begin{bmatrix}
-1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
$$

Matrix $\mathbf{b}$ is a diagonal matrix that contains the eigenvalues of $A$ on the diagonal. Matrix $\mathbf{a}$ consists of eigenvectors of $A$ (aligned as columns) corresponding to the eigenvalues on $\mathbf{b}$. For example, the vector

$$
\mathbf{v}_1 = \begin{bmatrix}
0 \\
0 \\
1 \\
0
\end{bmatrix}
$$

which is the first column of $\mathbf{a}$, is an eigenvector of $\lambda = -1$.

Vectors

$$
\mathbf{v}_2 = \begin{bmatrix}
0 \\
1 \\
0 \\
0
\end{bmatrix}
\quad \text{and} \quad
\mathbf{v}_3 = \begin{bmatrix}
0.7071 \\
0 \\
0 \\
-0.7071
\end{bmatrix}
$$

are also eigenvectors of $\lambda = -1$. Why does the entries of $\mathbf{v}_3$ look so ugly? This is because Matlab tries to rescale eigenvector to be of magnitude 1. The magnitude of $\mathbf{v}_3$ is

$$
\sqrt{(0.7071)^2 + 0^2 + 0^2 + (-0.7071)^2} = 1
$$

One can replace $\mathbf{v}_3$ by

$$
\mathbf{v}_3' = \begin{bmatrix}
1 \\
0 \\
0 \\
-1
\end{bmatrix}
$$

which is a nicer scaling of $\mathbf{v}_3'$. Continue next time.