Continue the example last time:

\[ f : M_{2 \times 2}(\mathbb{R}) \to M_{2 \times 2}(\mathbb{R}) \]

\[ f \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \]

Find the eigenvalues and eigenvectors of \( f \).

We translated the problem of finding eigenvalues and eigenvectors of \( f \) to the problem of finding the eigenvalues and eigenvectors of the matrix

\[
A = [f]_B = \begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix}
\]

We found that \( A \) has two eigenvalues \( \lambda = 1 \) and \( \lambda = -1 \).

The eigenspace of \( A \) corresponding to \( \lambda = 1 \) is

\[
\widetilde{E}_1 = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}
\]

The eigenspace of \( A \) corresponding to \( \lambda = -1 \) is

\[
\widetilde{E}_{-1} = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}
\]

Now we translate back to \( f \): the eigenvalues of \( f \) are \( \lambda = \pm 1 \).

The eigenspace of \( f \) corresponding to \( \lambda = 1 \) is

\[
E_1 = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}
\]

because \( \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \) is the vector in \( M_{2 \times 2}(\mathbb{R}) \) that has coordinate \( \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \) with respect to basis \( \mathcal{B} \).
Similarly, the eigenspace of $f$ corresponding to $\lambda = -1$ is
\[ E_{-1} = \text{span} \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}. \]

We have found the eigenvalues and eigenspaces of $f$. Our method was to translate the problem on the abstract vector space $M_{2 \times 2}(\mathbb{R})$ to a concrete vector space $\mathbb{R}^4$. This is done by fixing a basis of $M_{2 \times 2}(\mathbb{R})$ (we chose the standard basis) and replace abstract vectors (which are matrices in this case) by their coordinate vectors (vectors in $\mathbb{R}^4$).

There is another method to find the eigenvalues and eigenspaces of $f$ which doesn't resort to coordinates. We will discuss it after the midterm.