The addition and scaling operator of a general vector space are not sufficient to capture the idea of angles, which is a natural geometric of $\mathbb{R}^2$, $\mathbb{R}^3$ and $\mathbb{R}^n$ in general. To define angles, one needs to enrich the structure of a vector space. It turns out that we need something similar to the dot product in $\mathbb{R}^n$. Inner product is a generalization of dot product to a general vector space. Recall the definition of inner product:

Let $V$ be a vector space over $F$. A binary operator on $V$, taking value on $F$, is an inner product if it satisfies the following axioms:

1) Linearity in the first argument:
   \[(u + v, w) = (u, w) + (v, w), \quad \forall u, v, w \in V,\]
   \[(cu, w) = c(u, w), \quad \forall u, w \in V, \quad c \in F,\]

2) Conjugate symmetry:
   \[(u, v) = \overline{(v, u)} \quad \forall u, v \in V.\]

3) Positive definiteness:
   \[(u, u) > 0 \quad \forall u \in V.\]

If $(u, u) = 0$ then $u = 0$.

A vector space equipped with an inner product is called an inner product space.

One may ask: how about linearity in the second argument? Is it true? We in fact have the following:

\[(u, v + w) = (u, v) + (u, w), \quad \forall u, v, w \in V,\]
\[(u, cw) = \overline{c}(u, w), \quad \forall u, w \in V, \quad c \in F.\]
Why is this true?

\[
(u, v+w) = \overline{(v+w, u)} \quad \text{(conjugate symmetry)}
\]

\[
= \overline{(v, u)} + \overline{(w, u)} \quad \text{(linearity in the first argument)}
\]

\[
= (\overline{v}, \overline{u}) + \overline{(w, u)}
\]

\[
= (u, v) + (u, w).
\]

We see that the inner product is additive in the second argument. On the other hand,

\[
(u, cv) = \overline{(cv, u)} = \overline{c(v, u)} = \overline{c} \overline{(v, u)} = \overline{c} (u, v).
\]

We see that the inner product is almost scalar multiplicative on the second argument. When we factor a constant outside of the inner product, we have to take the conjugate of the constant factor.

An operator \((\cdot, \cdot)\) that satisfies the axioms (1) and (2) is called a sesquilinear form. The prefix sesqui means \(1\frac{1}{2}\). If the operator is linear in first and second argument, it would be called a bilinear form.

If \(F = \mathbb{R}\) then an inner product on \(V\) is a bilinear form. This is because \(\overline{c} = c\) for any \(c \in \mathbb{R}\).

\[Ex: \text{ Let } V \text{ be an inner product space. Show that } (0,0) = 0. \]

By the linearity on the first argument:

\[
(0,0) = (0+0, 0) = (0,0) + (0,0)
\]

Thus, \((0,0) = 0\).

\[Ex: \text{ On } \mathbb{C}, \text{ consider the operator } (\cdot, \cdot) \text{ given by } \]

\[
(\overline{z}, w) = \overline{zw} \quad \forall z, w \in \mathbb{C}
\]

Show that \((\cdot, \cdot)\) is an inner product on \(\mathbb{C}\).

We need to check all the axioms.

* Check linearity on the first argument:
Let $\bar{z}, w, v \in C$. We have
\[(z + v, w) = (\bar{z} + \bar{v}) \bar{w} = \bar{z} \bar{w} + \bar{v} \bar{w} \text{ (distribution rule)} = (z, w) + (v, w)\]

Let $z, w \in C$ and $c \in C$. We have
\[(cz, w) = c \bar{z} \bar{w} = c(z, w)\]  
* Check conjugate symmetry:

Let $z, w \in C$. We have
\[(z, w) = \bar{z} \bar{w} = \bar{w} \bar{z} = (w, z)\]

* Check positive definiteness:

Let $z \in C$. We have
\[(z, z) = z \bar{z} = |z|^2 \geq 0\]

Recall that if $z = a + ib$ (a, b \in R) then $\bar{z} = (a + ib)(a - ib) = a^2 + b^2 = |z|^2$.

If $(z, z) = 0$ then $a^2 + b^2 = 0$. This happens only if $a = b = 0$, which means $z = 0$.

Ex: On $C$, consider an operator $(\bar{z}, w) = \bar{w}$. Show that $(., .)$ is not an inner product.

To show that $(., .)$ is not an inner product, we only need to give a counterexample showing how one of the axioms is violated.

Let us take $z = i$. Then
\[(i, i) = (i, i) = i^2 = -1 < 0\]

Thus the positive definiteness condition is violated.

* Norm induced by an inner product:

Let $V$ be an inner product space. The operator $\| \cdot \|$ given by
\[\|v\| = \sqrt{(v, v)}\]

is called the norm on $V$ induced by (or associated with) the inner product.
Norm of a vector is also called length or magnitude of the vector.

**Ex:**

Let $V$ be a real inner product space (i.e. $F=\mathbb{R}$) and $u, v \in V$ be of equal length. Show that $u+v$ and $u-v$ are perpendicular to each other.

(Two vectors called perpendicular or orthogonal to each other if their inner product is equal to zero.)

We have $\|u\| = \|v\|$. Thus, $\sqrt{(u, u)} = \sqrt{(v, v)}$. We get $(u, u) = (v, v)$.

We want to show $(u+v, u-v) = 0$.

We have

$$
(u+v, u-v) = (u, u-v) + (v, u-v)
$$

$$
= (u, u) - (u, v) + (v, u) - (v, v)
$$

Because $(u, u) = (v, v)$, we get

$$(u+v, u-v) = -(u, v) + (v, u) \quad (\star)
$$

By the conjugate symmetry property of inner product,

$$
(u, v) = \overline{(v, u)} \quad (\star \star)
$$

Since $F=\mathbb{R}$, $(v, u) \in \mathbb{R}$. Thus, $(v, u) = (u, v)$. Then $(\star \star)$ becomes

$$
(u, v) = (u, v).
$$

Applying this identity to $(\star)$, we get $(u+v, u-v) = -(u, v) + (u, v) = 0$.

Therefore, $u+v$ and $u-v$ are orthogonal to each other.