Let $V$ be an inner product space. Let $u, v \in V$ and $a, b, c, d \in \mathbb{C}$.

How do we expand the inner product $(au + bv, cu + dv)$?

We have

$$(au + bv, cu + dv) = (au, cu + dv) + (bv, cu + dv)$$

(linearity in the first argument)

$$= (au, cu) + (au, du) + (bv, cu) + (bv, du)$$

(additive in the second argument)

$$= a\overline{c}(u, u) + a\overline{d}(u, v) + b\overline{c}(u, v) + b\overline{d}(u, u).$$

An easy way to remember this is to regard the inner product as a "regular" product $(au + bv)(cu + dv)$ although this product doesn't make sense. Then use distribution law:

$$ac u_i u_i + bc u_i u_i + ad u_i u_i + bd u_i u_i.$$ 

Each coefficient on the second factor has to come with complex conjugate:

$$a\overline{c}(u, u_i) + b\overline{c}(u, u_i) + a\overline{d}(u, u_i) + b\overline{d}(u, u_i).$$

This is the formula one needs for Problem 3 of HW 5.

**Ex:** (Pythagorean theorem)

Let $V$ be an inner product space. Let $u, v \in V$ be such that $\langle u, v \rangle = 0$. Show that

$$\|u + v\|^2 = \|u\|^2 + \|v\|^2. \quad (*)$$

We have

$$LHS(\ast) = (u + v, u + v)$$

$$= (u, u) + (v, u) + (u, v) + (v, v).$$

Because $u \perp v$, we have

$$\langle u, v \rangle = (u, v) = 0.$$ 

Thus, $LHS(\ast) = (u, u) + (v, v) = \|u\|^2 + \|v\|^2 = RHS(\ast)$. 


We know that inner product provides an additional structure to a vector space that helps us define angles and lengths (norms). There is another structure (weaker) that helps us define lengths, but not angles. That is norm.

**Definition:**

Let \( V \) be a vector space over \( F = \mathbb{R}, \mathbb{C} \). An operator \( \| \cdot \| : V \to \mathbb{R} \) is said to be a norm on \( V \) if it satisfies three following axioms:

1. **Homogeneity:**
   \[ \| cu \| = |c| \| u \| \quad \forall u \in V, c \in F. \]

2. **Triangle inequality:**
   \[ \| u + v \| \leq \| u \| + \| v \| \quad \forall u, v \in V. \]

3. **Positivity:**
   \[ \| u \| \geq 0 \quad \forall u \in V. \]
   
   If \( \| u \| = 0 \) then \( u = 0 \).

A vector space equipped with a norm is called a **normed space**.

Norm is a generalization of length in \( \mathbb{R}^n \) to an abstract vector space. All of the axioms reflect basic properties of length.

The homogeneity property says that if we scale a vector by a factor \( c \) then the length of \( u \) is scaled by factor \( |c| \).

The triangle inequality reflects a geometric inequality well-known in \( \mathbb{R}^n \).
If \( V \) is an inner space then \( \|u\| = \sqrt{u,u} \) defines a norm on \( V \). One can easily check that this satisfies all the axioms of norm. This norm is a special type of norm in that it comes from an inner product.

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vector space \rightarrow \text{inner product} \rightarrow \text{inner product space (angles, lengths)}
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Normed space (lengths)

Not all norms come from an inner product. Indeed, the norm induced from an inner product has to satisfy a so-called parallelogram identity.

Ex: Let \( V \) be an inner product. Then
\[
\|u + v\|^2 + \|u - v\|^2 = 2(\|u\|^2 + \|v\|^2) \quad \forall u, v \in V.
\]

```
\[\begin{array}{c}
\text{v} \\
\text{u+v} \\
\text{u-v} \\
\text{u} \\
\text{0}
\end{array}\]
```

The sum of the square of the diagonals is equal to the sum of the square of the edges.

Why is this true?
\[
\|u + v\|^2 = (u + v, u + v) = (u, u) + 2(u, v) + (v, v),
\]
\[
\|u - v\|^2 = (u - v, u - v) = (u, u) - 2(u, v) + (v, v).
\]

Sum these two equations:
\[
\|u + v\|^2 + \|u - v\|^2 = 2(u, u) + 2(v, v) = 2\|u\|^2 + 2\|v\|^2.
\]

Ex: \( \mathbb{R}^2 \) has a norm \( \|x\| = \sqrt{x_1^2 + x_2^2} \) where \( x = [x_1 \ x_2] \).

Why is this true?
We need to check every axiom of norm.
* Check homogeneity:
Let $c \in \mathbb{R}$ and $x \in \mathbb{R}^2$. We want to show that $\|cx\| = |c| \|x\|$. 
Write $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Then $cx = c \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} cx_1 \\ cx_2 \end{bmatrix}$.

Then $\|cx\| = |cx_1| + |cx_2| = |c| |x_1| + |c| |x_2| = |c| (|x_1| + |x_2|) = |c| \|x\|$,.

* Check triangle inequality:
We know the triangle inequality of numbers:
$|a + b| \leq |a| + |b| \quad \forall a, b \in \mathbb{R}$. (*)

Let $x, y \in \mathbb{R}^2$. We want to show $\|x + y\| \leq \|x\| + \|y\|$.

We write $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$.

Then $x + y = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \end{bmatrix}$.

Then $\|x + y\| = |x_1 + y_1| + |x_2 + y_2| 
\leq (|x_1| + |y_1|) + (|x_2| + |y_2|) \quad \text{(due to (*) )} 
= (|x_1| + |x_2|) + (|y_1| + |y_2|) 
= \|x\| + \|y\|.$

* Check positivity: (can be done easily).

Ex: The norm $\| \cdot \|$ on $\mathbb{R}^2$ given by $\|x\| = |x_1| + |x_2|$ doesn’t come from any inner product.
If this norm comes from an inner product then it must satisfy the parallelogram identity. Let us pick 
$x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $y = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$.

Then $x + y = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $x - y = \begin{bmatrix} 0 \\ 5 \end{bmatrix}$.
Then \[ \|x+y\| = |12| + |11| = 3, \]
\[ \|x-y\| = |19| + |15| = 5, \]
\[ \|x\| = |11| + |12| = 3, \]
\[ \|y\| = |11| + |13| = 4. \]

We see that
\[
\begin{align*}
\|x+y\|^2 + \|x-y\|^2 &= 3^2 + 5^2 = 34 \\
2(\|x\|^2 + \|y\|^2) &= 2(3^2 + 4^2) = 50. 
\end{align*}
\]

Because the parallelogram law is not satisfied, \(\|\cdot\|\) doesn’t come from an inner product.

This norm is known as the taxi cab norm. Imagine a taxi going from point \(O\) (the origin) to a point \(A\) on the plane. This point corresponds to vector
\[
x = \overrightarrow{OA} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.
\]

The taxi only travel on the roads, which only go horizontally or vertically. The shortest path to go from \(O\) to \(A\) is \(|x_1| + |x_2|\).