An application of inner product is to define projections. Let $V$ be an inner product space.

**Definition:**
- Two vectors $u$ and $v$ are said to be perpendicular (or orthogonal) to each other (denoted by $u \perp v$) if $(u, v) = 0$.
- Let $u \in V$ and $E$ be a subspace of $V$. Then $u$ is said to be perpendicular (or orthogonal) to $E$ (denoted by $u \perp E$) if $u$ is perpendicular to every vector in $E$.

This definition is quite natural. We know from geometry that a vector is said to be perpendicular to a plane if it is perpendicular to every vector on the plane. The definition we just gave is a generalization of this concept. One now can talk about a vector being orthogonal to a subspace, not just a 2D-plane.

In 3D-geometry, one can talk about orthogonal projection of a vector on a plane. We can generalize this idea as follows.

**Definition:**
- Vector $v$ is said to be the orthogonal projection of $u$ on subspace $E$ if two following conditions are satisfied:
  1. $v \in E$,
  2. $u - v \perp E$.

The question now is how to compute the projection of $u$ on $E$. In other words, given $u$ and $E$, we want to solve for vector $v \in E$ such that $u - v \perp E$. We acknowledge a difficulty; it is practically
To check if a vector is perpendicular to every single vector in a vector space $E$, a vector space has infinitely many vectors! However, we in fact only need to check if the vector is perpendicular to every vector in a basis. The following theorem explains this idea.

**Theorem:**

Let $V$ be an inner product, $E$ be a subspace of $V$, and $u \in V$.

Let $\mathcal{B} = \{v_1, v_2, \ldots, v_n\}$ be a basis of $E$. Then $u \perp E$ if and only if $u \perp v_k$ for all $k=1,2,\ldots,n$.

Why is this true? The theorem says "if and only if". So there are two statements to check:

(a) Suppose $u \perp E$. Show that $u \perp v_k$ for all $k=1,2,\ldots,n$.

(b) Suppose $u \perp v_k$ for all $k=1,2,\ldots,n$. Show that $u \perp E$.

Part (a) is quite easy! Because $u \perp E$, $u$ has to be perpendicular to every vector in $E$. Since $v_1, v_2, \ldots, v_n$ are vectors in $E$, $u$ has to be perpendicular to each of them.

Part (b) is more interesting to prove. We have

$$u \perp v_k \quad \forall k=1,2,\ldots,n.$$

We want to show $u \perp E$. That is to show $u \perp v \quad \forall v \in E$.

Let $v \in E$. We want to show $u \perp v$. That is to show $(u,v) = 0$.

Because $v_1, v_2, \ldots, v_n$ form a basis of $E$, $v$ is a linear combination of these vectors. We can write

$$v = q_1 v_1 + q_2 v_2 + \ldots + q_n v_n,$$

for some $q_1, q_2, \ldots, q_n \in \mathbb{F}$. Then

$$(u,v) = (u, q_1 v_1 + q_2 v_2 + \ldots + q_n v_n)$$

$$= (u, q_1 v_1) + (u, q_2 v_2) + \ldots + (u, q_n v_n)$$

$$= \underbrace{q_1 (u,v_1)}_{0} + \underbrace{q_2 (u,v_2)}_{0} + \ldots + \underbrace{q_n (u,v_n)}_{0} = 0.$$
Thus \((u, v) = 0\). The theorem is proven.

We return to the question we asked earlier: given a vector \(u \in V\) and a subspace \(E \subseteq V\), how do we find the projection of \(u\) onto \(E\)? Let this projection be \(v \in E\). We can write \(v\) as a linear combination of \(v_1, v_2, \ldots, v_n\):

\[
v = c_1 v_1 + c_2 v_2 + \ldots + c_n v_n.
\]

We want to solve for \(c_1, c_2, \ldots, c_n \in E\) such that \(u - v \perp E\).

The condition \(u - v \perp E\) boils down to the condition

\[
(u - v, v) = (u - v, v_1) = \ldots = (u - v, v_n) = 0.
\]

The first term is equal to

\[
(u - v, v) = (u, v) - (v, v)
\]

\[
= (u, v) - (c_1 v_1 + c_2 v_2 + \ldots + c_n v_n, v_1)
\]

\[
= (u, v) - c_1 (v_1, v_1) - c_2 (v_2, v_1) - \ldots - c_n (v_n, v_1).
\]

Since this is supposed to be zero, we get an equation

\[
g_1 (v_1, v_1) + g_2 (v_2, v_1) + \ldots + g_n (v_n, v_1) = (u, v_1),
\]

\[
\text{known} \quad \text{known} \quad \text{known} \quad \text{known}
\]

Similarly, the equation \((u - v, v_2) = 0\) gives us an equation

\[
g_1 (v_1, v_2) + g_2 (v_2, v_2) + \ldots + g_n (v_n, v_2) = (u, v_2),
\]

\[
\text{known} \quad \text{known} \quad \text{known} \quad \text{known}
\]

And so on. We eventually get a linear system of \(n\) equations and \(n\) unknowns.

If \(B\) is orthogonal, i.e., any two vectors in \(B\) are perpendicular to each other, then the first equation becomes

\[
g_1 (v_1, v_1) = (u, v_1).
\]

The second equation becomes

\[
g_2 (v_2, v_2) = (u, v_2).
\]
And so on. In this case, the system becomes trivial to solve.

**Definition:** A set of vectors \( \{v_1, v_2, \ldots, v_n\} \) is said to be orthogonal if any two vectors in this set are perpendicular to each other: \( v_j \perp v_k \) for any \( j \neq k \). If the set is orthogonal and \( \|v_1\| = \|v_2\| = \cdots = \|v_n\| = 1 \) then it is said to be orthonormal.

From our analysis above, if \( E \) has an orthogonal basis \( B = \{v_1, v_2, \ldots, v_n\} \) then the projection of \( u \in V \) on \( E \) is given by

\[
v = \text{proj}_E u = \frac{(u, v_1)}{\|v_1\|^2} v_1 + \frac{(u, v_2)}{\|v_2\|^2} v_2 + \cdots + \frac{(u, v_n)}{\|v_n\|^2} v_n.
\]

Note that this formula is true only if \( B \) is an orthogonal basis.

Sometimes, for convenience we write \( \text{proj}_{\{v_1, \ldots, v_n\}} u \) instead of \( \text{proj}_E u \).

The notation \( \text{proj}_{\{v_1, \ldots, v_n\}} u \) denotes the projection of vector \( u \) on the vector space spanned by \( \{v_1, \ldots, v_n\} \).

**Ex:** Let \( a \in V \) and \( u \in V \). What is the projection of vector \( u \) on the line that contains \( a \)?

The line is the vector space spanned by the vector \( a \). It has basis \( B = \{a\} \). This is an orthogonal basis since it contains only one element. Thus,

\[
v = \text{proj}_{\{a\}} u = \frac{(u, a)}{\|a\|^2} a.
\]