Recall that the vector space \( V = \{ u: C[0,1] \to \mathbb{R}, u \text{ is continuous} \} \) is an inner product space with inner product defined by

\[
(u, v) = \int_0^1 u(x) v(x) \, dx.
\]

This inner product induces a norm

\[
|u|_V\sqrt{u} = \left( \int_0^1 u^2(x) \, dx \right)^{1/2}.
\]

This norm is sometimes referred to as an energy norm. When \( u \) is a velocity function then \(|u|_V^2\) is proportional to the energy.

The problem of finding a parabola \( v(x) = ax^2 + bx + c \) that best approximates the cosine function \( u(x) = \cos x \) becomes the problem of finding the projection \( v = \text{proj}_{P_2(\mathbb{R})} u \).

We know two ways to do this: solving a system of equations or orthogonalizing a basis. Let us choose the first way.

A basis of \( P_2(\mathbb{R}) \) is \( B = \{ x^0, x, 1 \} \). Then \( v = c_0 u_0 + c_1 u_1 + c_3 u_3 \)

where \( c_1, c_2, c_3 \) solve the matrix equation

\[
\begin{bmatrix}
(u_1, u_1) & (u_1, u_2) & (u_1, u_3) \\
(u_2, u_1) & (u_2, u_2) & (u_2, u_3) \\
(u_3, u_1) & (u_3, u_2) & (u_3, u_3)
\end{bmatrix}
\begin{bmatrix}
c_1 \\
c_2 \\
c_3
\end{bmatrix}
= \begin{bmatrix}
(u, u_1) \\
(u, u_2) \\
(u, u_3)
\end{bmatrix}
\]

We have

\[
(u_1, u_1) = (x^0, x^0) = \int_0^1 x^0 \, dx = \frac{1}{5},
\]

\[
(u_2, u_1) = (x, x^0) = \int_0^1 x \, dx = \frac{1}{2},
\]

\[
(u_3, u_1) = (1, x^0) = \int_0^1 1 \, dx = 1.
\]
\[
(u_3, u_1) = (1, x^3) = \int_0^1 x^3 \, dx = \frac{1}{3},
\]
\[
(u_2, u_2) = (2, x^2) = \int_0^1 x^2 \, dx = \frac{1}{3},
\]
\[
(u_1, u_1) = (1, 1) = \int_0^1 1 \, dx = 1.
\]
\[
(u_2, u_1) = \int_0^1 (\cos x) x^2 \, dx = 2 \cos 1 - \sin 1.
\]
(Integration by parts twice)
\[
(u_2, u_2) = \int_0^1 (\cos x) x \, dx = \sin 1 + \cos 1 - 1
\]
\[
(u_1, u_3) = \int_0^1 \cos x \, dx = \sin 1.
\]

The matrix equation now becomes
\[
\begin{bmatrix}
u_3 & 1/4 & 1/3 \\
1/4 & 1/3 & 1/2 \\
1/3 & 1/2 & 1
\end{bmatrix}
\begin{bmatrix}
c_1 \\
c_2 \\
c_3
\end{bmatrix} =
\begin{bmatrix}
2 \cos 1 - \sin 1 \\
\sin 1 + \cos 1 - 1 \\
\sin 1
\end{bmatrix}
\]

Then
\[
\begin{bmatrix}
c_1 \\
c_2 \\
c_3
\end{bmatrix} = A^{-1}b \approx
\begin{bmatrix}
-0.4310 \\
-0.0365 \\
1.0034
\end{bmatrix}
\]

(One can use the command \texttt{inv(A)*b} in Matlab.)

Therefore, the parabola that best approximates the cosine function on the interval \([0, 1]\) is

\[
\nu(x) = c_1 u_1 + c_2 u_2 + c_3 u_3
\]
\[
\approx -0.4310 x^3 - 0.0365 x + 1.0034.
\]
Ex: Find an orthonormal basis of $P_2(\mathbb{R})$. The inner product is still given by $\langle u, v \rangle = \int_0^1 u(x) v(x) \, dx$.

A basis of $P_2(\mathbb{R})$ is $B = \{ x^2, x, 1 \}$.

We use Gram-Schmidt procedure to orthogonalize $B$:

$v_1 = u_1 = x^2$,

$v_2 = u_2 - \frac{\langle u_2, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 = u_2 - \frac{\langle u_2, 1 \rangle}{\langle 1, 1 \rangle} v_1$.

Since $\langle u_2, v_1 \rangle = \int_0^1 x^3 \, dx = \frac{1}{4}$ and $\langle v_1, v_1 \rangle = \int_0^1 x^4 \, dx = \frac{1}{5}$,

we get $v_2 = x - \frac{\frac{1}{4}}{\frac{1}{5}} x^2 = x - \frac{5}{4} x^2$.

Then $v_3 = u_3 - \frac{\langle u_3, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 - \frac{\langle u_3, v_2 \rangle}{\langle v_2, v_2 \rangle} v_2$.

We have $\langle u_3, v_1 \rangle = \int_0^1 x^2 \, dx = \frac{1}{3}$

$\langle u_3, v_2 \rangle = \int_0^1 x \, dx = \frac{1}{2}$

$\langle v_2, v_2 \rangle = \int_0^1 \left(x - \frac{5}{4} x^2\right)^2 \, dx = \frac{1}{4}$

Thus, $v_3 = 1 - \frac{\frac{1}{3}}{\frac{1}{5}} x^2 - \frac{\frac{1}{2}}{\frac{1}{4}} \left(x - \frac{5}{4} x^2\right)$

$= 1 - \frac{10}{3} x^2 + \frac{5}{8} x^2$.

We obtain an orthogonal basis of $P_2(\mathbb{R})$, namely $B = \{ v_1, v_2, v_3 \}$.

To get an orthonormal basis of $B$, we rescale $v_1, v_2, v_3$ to make sure that they have length equal to 1.
\[
\omega_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{1/5}} \quad v_1 = \frac{1}{\sqrt{1/5}} x^2 = \sqrt{5} x^2,
\]

\[
\omega_2 = \frac{v_2}{\|v_2\|} = \frac{1}{\sqrt{\langle v_2, v_2 \rangle}} \quad \omega_2 = \frac{1}{\sqrt{1/3}} \left( x - \frac{5x^3}{4} \right) = \sqrt{3} \left( x - \frac{5x^3}{4} \right),
\]

\[
\omega_3 = \frac{v_3}{\|v_3\|} = \ldots
\]

Then \( \{\omega_1, \omega_2, \omega_3\} \) is an orthonormal basis of \( L_2(\mathbb{R}) \).