Consider the set $V = \left\{ \begin{bmatrix} a & b \\ i(ax + b) & 0 \end{bmatrix} : a, b \in \mathbb{R} \right\}$.

One can see that $V$ is a subset of $M_{2\times2}(\mathbb{C})$.

However, $V$ is not a subspace of $M_{2\times2}(\mathbb{C})$ when viewed as a vector space over $\mathbb{C}$. In other words, $V$ is not a vector space over $\mathbb{C}$. The reason is that $V$ is not closed under scaling by a complex number. For example, the matrix

$$A = \begin{bmatrix} 1 & 0 \\ i & 0 \end{bmatrix}$$

belongs to $V$ (with $a = 1$, $b = c = 0$), but the matrix

$$iA = \begin{bmatrix} i & 0 \\ -i & 0 \end{bmatrix}$$

doesn't belong to $V$.

Nevertheless, $V$ is a vector space over $\mathbb{R}$. How so? We have a general rule as follows.

*Theorem.*

If $V$ is a vector space over $\mathbb{C}$, then it is also a vector space over $\mathbb{R}$ and over $\mathbb{Q}$. If $V$ is a vector space over $\mathbb{R}$, then it is also a vector space over $\mathbb{Q}$. In short, if $V$ is a vector space over a big field, then it is a vector space over smaller fields.

$$\mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$$

We know that $M_{2\times2}(\mathbb{C})$ is a vector space over $\mathbb{C}$. Therefore, it is also a vector space over $\mathbb{R}$. One can show that $V$ is a subspace of $M_{2\times2}(\mathbb{C})$ (when viewed as a vector space over $\mathbb{R}$) by checking that

1. $0 \in V$,
2. $V$ is closed under addition,
3. $V$ is closed under scaling (by real numbers).
What is a basis and the dimension of $V$?

We can rewrite $V$ as

$$V = \{ a \begin{bmatrix} 1 & 0 \\ i & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ i & 0 \end{bmatrix} : a, b, c \in \mathbb{R} \}$$

$$= \text{span} \left\{ \begin{bmatrix} 1 & 0 \\ i & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ i & 0 \end{bmatrix} \right\}$$

Let $B = \{ A_1, A_2, A_3 \}$. We see that $V = \text{span} B$. To say that $B$ is a basis of $V$, one needs to check if $B$ is linearly independent.

Consider the equation $c_1 A_1 + c_2 A_2 + c_3 A_3 = 0$ with unknowns $c_1, c_2, c_3 \in \mathbb{R}$.

This equation is equivalent to

$$\begin{bmatrix} c_1 & c_2 \\ i(c_1 + c_2) & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Thus, $c_1 = c_2 = c_3 = 0$. We conclude that $B$ is a basis of $V$. Moreover,

$$\text{dim}_{\mathbb{R}} V = 3.$$

* Linear maps.

Let $V$ and $W$ be vector spaces over $F$. There are many maps from $V$ to $W$. Think of the case $V = W = \mathbb{R}$. There are a lot of maps from $\mathbb{R}$ to $\mathbb{R}$. We will be considering a useful type of maps called linear maps.

A map $f : V \to W$ is said to be linear (over $F$) if it satisfies two following properties:

1. **Additive:**

   $$f(u + w) = f(u) + f(w) \quad \forall u, w \in V$$

   (“First add, then apply $f$” is equal to “first apply $f$, then add.”)
2) Scalar multiplicative:

\[ f(cv) = cf(v) \quad \forall c \in \mathbb{C}, v \in V. \]

("First scale, then apply \( f \) is the same as "first apply \( f \), then scale.")

* Most maps are not linear, for example \( f(x) = x^2 \) (violate additive rule), \( f(x) = \sin x \) (violate both additive and scaling rule),...

The set of all linear maps from \( V \) to \( W \) is denoted as \( \mathcal{L}(V, W) \).

* A useful fact about linear maps is that they always map to zero vector to the zero vector. To see why, one can apply the additive rule for \( v = w = 0 \):

\[ f(0 + 0) = f(0) + f(0) \]

This implies \( f(0) = f(0) + f(0) \). By the Cancellation Law (Homework 1), we obtain \( f(0) = 0 \).

**Ex:** Let \( V \) be the set of all smooth functions from \((0,1)\) to \( \mathbb{R} \).

By smooth, we mean infinitely differentiable. One can check without difficulty that \( V \) is a vector space over \( \mathbb{R} \).

The differential operator \( D : V \rightarrow V, D(u) = u' \) is a linear map. This is because

\[ (u+v)' = u' + v' \]
\[ (cu)' = cu' \]

**Ex:** Let \( V \) be the space of all continuous functions from \([0,1]\) to \( \mathbb{R} \).

The integral operator \( I : V \rightarrow \mathbb{R}, I(u) = \int_0^1 u(x) \, dx \)

is a linear map. This is because

\[ \int_0^1 (u(x)+v(x)) \, dx = \int_0^1 u(x) \, dx + \int_0^1 v(x) \, dx, \]
\[ \int_0^1 cu(x) \, dx = c \int_0^1 u(x) \, dx \]
Ex: The determinant map $\det : M_{2 \times 2}(\mathbb{C}) \to \mathbb{C}$,

$$\det \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = ad - bc$$

is not linear because it violates the addition rule:

$$\det \left( 2 \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \right) = \det \left( \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix} \right) = -8 \quad \text{different}$$

$$2 \cdot \det \left( \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \right) = 2 \cdot (4 - 6) = -4$$

Ex: Problem 2 on the worksheet.

$f : \mathbb{C} \to \mathbb{C}$, $f(x) = \overline{x}$

Recall: if $z = a + bi$ then $\overline{z} = a - bi$.

(a) Show that $f$ is a linear map over $\mathbb{R}$.

We need to check two properties:

• Check if $f$ is additive:
  That is to check $f(x + y) = f(x) + f(y)$ \forall x, y \in \mathbb{C}$
  Let $x_1, x_2 \in \mathbb{C}$. We want to show
  \[ f(x_1 + x_2) = f(x_1) + f(x_2) \]
  That is to show
  \[ \overline{x_1 + x_2} = \overline{x_1} + \overline{x_2}. \]
  This is a well-known property of complex numbers.

• Check if $f$ is scalar multiplicative (over $\mathbb{R}$):
  That is to check $f(cx) = cf(x)$ \forall c \in \mathbb{R}, x \in \mathbb{C}$.
  Let $c \in \mathbb{R}$, $x \in \mathbb{C}$. We want to show
  \[ f(cx) = cf(x) \]
  That is to show
  \[ \overline{c \cdot x} = c \cdot \overline{x}. \]
  This is a well-known property of complex numbers.
(b) $f$ is not a linear map over $\mathbb{C}$ because it violates the scalar multiplication rule. For example,

$$f(i \cdot 1) = f(i) = -i \neq i = i \cdot 1$$