let us continue the example last time:
\[ G : \mathbb{P}_2(\mathbb{R}) \rightarrow \mathbb{P}_2(\mathbb{R}) \]
\[ G(u) = (x+1)u' \]

Find a basis and the dimension of range \((G)\)

By definition of the range space,
\[ \text{range } G = \{ G(u) : u \in \mathbb{P}_2(\mathbb{R}) \} \]
\[ = \{ (x+1)u' : u = ax^2 + bx + c, a, b, c \in \mathbb{R} \} \]
\[ = \{ (x+1)(ax + b) : a, b, c \in \mathbb{R} \} \]
\[ = \{ 2a(x^2 + x) + b(x+1) : a, b \in \mathbb{R} \} \]
\[ = \text{span}\{ x^2 + x, x + 1 \}. \]

To conclude that \(\{u_1, u_2\}\) is a basis of range \((G)\), we need to check if it is linearly independent. Consider the equation
\[ G(x^2 + x) + c_2(x+1) = 0 \]
with unknowns \(c_1, c_2 \in \mathbb{R}\). The equation has to be true for all \(x \in \mathbb{R}\).
One can see that there are infinitely many equations while only two unknowns. It would not be surprising that \(c_1\) and \(c_2\) have to be equal to zero. Pick \(x = 1\), we get
\[ 2c_1 + 2c_2 = 0 \]
or simply \(c_1 + c_2 = 0\). Pick \(x = 0\), we get \(c_2 = 0\). Therefore, \(c_1 = c_2 = 0\).
We conclude that \(\{u_1, u_2\}\) is indeed a basis of range \((G)\).
\[ \text{rank}(G) = 2 \]

* Relation of null space and range space:

Let \(f : V \rightarrow W\) be a linear map. Let
\[ B_1 = \{v_1, v_2, \ldots, v_m\}, \]
\[ B_2 = \{w_1, w_2, \ldots, w_n\} \]
be bases of \(V\) and \(W\) respectively.
The matrix representation of \( f \) is
\[
A = \begin{bmatrix} [f(v_1)]_{B_2} & [f(v_2)]_{B_2} & \cdots & [f(v_n)]_{B_2} \end{bmatrix}_{m \times n}
\]

Matrix \( A \) can be regarded as a linear map "\( A \)" from \( \mathbb{R}^n \) to \( \mathbb{R}^m \), which does the following: it maps each \( x \in \mathbb{R}^n \) to vector \( Ax \in \mathbb{R}^m \).

The column space of matrix \( A \) is defined as the space spanned by the columns of \( A \). If we write
\[
A = \begin{bmatrix} C_1 & C_2 & \cdots & C_n \end{bmatrix}
\]

Then \( \text{col}(A) = \text{span} \{ C_1, C_2, \ldots, C_n \} \)

Note that one can write
\[
a_1C_1 + a_2C_2 + \cdots + a_nC_n = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \begin{bmatrix} C_1 & C_2 & \cdots & C_n \end{bmatrix} = Ax
\]
Thus, \( \text{col}(A) = \{Ax : x \in \mathbb{R}^n\} \).

This is exactly the range of the linear map \( A \). The column space of \( A \), understood as a matrix, is equal to the range of \( A \), understood as a linear map.

How about the row space of \( A \)? The row space is contained in the domain (\( \mathbb{R}^m \)), while the column space is contained in the target set (\( \mathbb{R}^n \)).

\[
A = \begin{bmatrix}
R_1 \\
R_2 \\
\vdots \\
R_m
\end{bmatrix}
\]

By definition of row space,

\( \text{row}(A) = \text{span}\{R_1, R_2, \ldots, R_m\} \)

We will see that the row space is perpendicular to the null space of \( A \).

To see this, we take an \( x \in \text{null} A \). We know that \( Ax = 0 \). In other words,

\[
\begin{bmatrix}
R_1 \\
R_2 \\
\vdots \\
R_m
\end{bmatrix}
\begin{bmatrix}
x \\
1
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
\vdots \\
0
\end{bmatrix}
\]

Hence, \( R_1 x = R_2 x = \ldots = R_m x = 0 \). In other words, \( x \) is perpendicular to vectors \( R_1, R_2, \ldots, R_m \). Thus, \( x \) is perpendicular to row(\( A \)).

Because \( x \) was taken arbitrarily in null(\( A \)), we conclude that \( \text{null} A \perp \text{row} A \).
Every vector on $\text{null} A$ will be mapped to 0. (i.e $\text{null} A$ is collapsed by $A$). If the linear map $A$ is restricted on $\text{row} A$ then the map will be one-to-one and onto. The row space and the column space are of the same “size”. To be more specific, 
$$\dim \text{row}(A) = \dim \text{col}(A).$$
This number is called the rank of $A$.

There is an important result in Linear Algebra called rank-nullity theorem. Recall the rank-nullity theorem for matrices states that:

Let $A$ be a matrix (not necessarily square). Then 
$$\text{rank}(A) + \text{nullity}(A) = \# \text{ columns of } A.$$ 

One can translate this theorem in terms of linear maps as follows. 

Let $f : V \rightarrow W$ be a linear map. Then 
$$\text{rank}(f) + \text{nullity}(f) = \dim V.$$ 

This theorem gives a way to compute rank through nullity and vice versa. If the rank of $f$ is large ($f$ is ‘rich’ in values) then the nullity is small ($f$ diminishes less).

Recall that the rank of a matrix is the number of nonzero rows (or equivalently, the number of pivot columns) in its RREF.
One can compute the rank and nullity through RRBF. For example,

\[
A = \begin{bmatrix}
1 & 2 & 3 & 0 \\
4 & 5 & 6 & 0 \\
7 & 8 & 9 & 0
\end{bmatrix}
\]

On Matlab:

\[
\text{>> A} = [1 2 3 0; 4 5 6 0; 7 8 9 0]
\]

\[
\text{>> rref(A)}
\]

The output is

\[
\begin{bmatrix}
1 & 0 & -1 & 0 & 1 \\
0 & 1 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

We see that rank(A) = 2. By rank-nullity theorem,

\[
\text{nullity}(A) = 4 - \text{rank}(A) = 2.
\]