A curve on the 2-dimensional plane can be regarded as a continuous map $f: \mathbb{R} \to \mathbb{R}^2$. There is a curve that fills the entire plane, known as a space-filling curve. This is an interesting object in a field of topology. A space-filling curve is an onto continuous map from $\mathbb{R}$ to $\mathbb{R}^2$. It is hard to visualize such a curve because it is pathological in many ways. For example, the curve is not a simple curve. In fact, it intersects itself infinitely many times.

However, one cannot find any map $f: \mathbb{R} \to \mathbb{R}^2$ that is both linear and onto (epimorphic). This will be explained by the rank-nullity theorem.

Let $f: V \to W$ be a linear map. The rank-nullity theorem says that

$$\text{rank}(f) + \text{nullity}(f) = \dim V.$$

Consequently, the following statements are equivalent:

- $f$ is monomorphic
- $\text{nullity}(f) = 0$
- $\text{rank}(f) = \dim V$

The following statements are also equivalent:

- $f$ is epimorphic
- $\text{rank}(f) = \dim W$
- $\text{nullity}(f) = \dim V - \dim W$
Let us consider 3 situations:

1. \( \dim V < \dim W \):

   In this case, \( f \) maps a small vector space into a large vector space. Intuitively, one can expect that \( f \) is not epimorphic. This can be proved as follows:

   \[
   \dim \text{range}(f) = \text{rank}(f) = \dim V - \text{nullity}(f) \\
   \leq \dim V \\
   < \dim W.
   \]

   Thus, \( \text{range}(f) \) is strictly smaller than \( W \). Therefore, \( f \) is not epimorphic.

   In particular, there is no linear map \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) that is onto.

   “One cannot cover a big bed with a small blanket.”

2. \( \dim V > \dim W \):

   In this case, \( f \) maps a big space into a smaller vector space. One can expect that \( f \) is not one-to-one. This observation can be proved as follows:

   \( \text{rank}(f) \leq \dim W < \dim V \).

   Thus,

   \[ \text{nullity}(f) = \dim V - \text{rank}(f) \geq 1. \]

   “One cannot fit a big blanket into a suitcase without folding it first.”

\[ V \to \quad \text{same value} \quad \to \quad \text{same value} \]
dim V = dim W:

Intuitively, dim V = dim W. By rank-nullity theorem, we observe that

\[ \text{nullity}(f) = 0 \iff \text{rank}(f) = n \iff f \text{ is isomorphic.} \]

Thus, in this case, “monomorphic”, “epimorphic”, and “isomorphic” are the same.

We summarize our above observations as follows:

*Theorem:*

Let \( f: V \to W \) be a linear map.

1. If \( \dim V < \dim W \) then \( f \) is not monomorphic.
2. If \( \dim V > \dim W \) then \( f \) is not epimorphic.
3. If \( \dim V = \dim W \) then \( f \) is monomorphic if and only if it is epimorphic.

Sum of two vector spaces

Consider two vector spaces \( U \) and \( V \) (over the same field \( F \)). It is easy to check that \( U \cap V \) is also a vector space.

Consider the intersection of \( U \) and \( V \)

How to check? Observe that \( U \cap V \) is a subset of \( U \), which is a vector space. One only needs to check 3 properties:

\[
\begin{align*}
\text{exercise} \begin{cases}
(1) & 0 \in U \cap V, \\
(2) & U \cap V \text{ is closed under addition,} \\
(3) & U \cap V \text{ is closed under scaling.}
\end{cases}
\end{align*}
\]
Ex: In $\mathbb{R}^2$, consider two lines passing through the origin. Each line can be viewed as a 1-dimensional vector space. We see from the picture that $U \cap V = \{0\}$, which is a vector space.

Ex: In $\mathbb{R}^3$, consider two planes that pass through the origin. The intersection of these planes is a line passing through the origin. Thus, $U \cap V$ is also a vector space.

While the intersection $U \cap V$ is always a vector space, the union $U \cup V$ is generally not a vector space. A simple example is that:

- $U$ and $V$ are two lines on the plane.
- The union of two lines is not a vector space because it is not closed under addition.

While the intersection $U \cap V$ is always a vector space, the union $U \cup V$ is generally not a vector space. A simple example is that:

- $U$ and $V$ are two lines on the plane.
- The union of two lines is not a vector space because it is not closed under addition.

A natural question is: what is a vector space that contains both $U$ and $V$?

The union $U \cup V$ is not an answer because it is not a vector space. There are in fact infinitely many vector spaces that contain both $U$
and $V$. For example, when $U$ and $V$ are lines, the plane that contains both lines is such a vector space. The 3-dimensional space as in the picture is also a vector space that contains both $U$ and $V$.

We therefore adjust the question to make it more meaningful:

What is the smallest vector space that contains both $U$ and $V$?

This vector space will be denoted as $U + V$ (the sum of vector space $U$ and vector space $V$). It is defined as

$$U + V = \{ u + v : u \in U, v \in V \}.$$

**Ex:**

$U + V =$ plane that contains both lines $U$ and $V$.

**Ex:**

$U = \text{xy-plane}$  
$V = \text{line passing through the origin}$  
$U + V = \mathbb{R}^3.$

How to find a basis of $U + V$?

Let $B_U$ be a basis of $U$, and $B_V$ be a basis of $V$. We know that
\[ U = \text{span} \mathbf{B}_1 \quad \text{and} \quad V = \text{span} \mathbf{B}_2 \]

Write

\[ \mathbf{B}_1 = \{ \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \}, \]
\[ \mathbf{B}_2 = \{ \mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m \}. \]

The concatenation of \( \mathbf{B}_1 \) and \( \mathbf{B}_2 \) is defined as

\[ \mathbf{B}_1 \sqcup \mathbf{B}_2 = \{ \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n, \mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m \}. \]

There is a slight difference between concatenation and union. If there are common vectors between \( \mathbf{B}_1 \) and \( \mathbf{B}_2 \), then the union \( \mathbf{B}_1 \cup \mathbf{B}_2 \) includes each of these vectors only once, while the concatenation includes these vectors twice. For example,

\[ \mathbf{B}_1 = \{ 1, 2, 3 \} \]
\[ \mathbf{B}_2 = \{ 3, 4, 5 \} \]
\[ \mathbf{B}_1 \cup \mathbf{B}_2 = \{ 1, 2, 3, 4, 5 \} \]
\[ \mathbf{B}_1 \sqcup \mathbf{B}_2 = \{ 1, 2, 3, 3, 4, 5 \}. \]

We see that the span of \( \mathbf{B}_1 \sqcup \mathbf{B}_2 \) includes both \( U \) and \( V \). Moreover, any vector space that contains both \( U \) and \( V \) must also contain \( \mathbf{B}_1 \sqcup \mathbf{B}_2 \). Thus, \( \text{span}(\mathbf{B}_1 \sqcup \mathbf{B}_2) \) is the smallest vector space that contains both \( U \) and \( V \). We get

\[ U + V = \text{span} (\mathbf{B}_1 \sqcup \mathbf{B}_2) \]

To find a basis of \( U + V \), we only need to extract linearly independent vectors from \( \mathbf{B}_1 \sqcup \mathbf{B}_2 \).

We do so by arranging vectors \( \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n, \mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m \) as columns of a matrix.

\[
\begin{bmatrix}
\mathbf{v}_1 & \mathbf{v}_2 & \ldots & \mathbf{v}_n & \mathbf{w}_1 & \mathbf{w}_2 & \ldots & \mathbf{w}_m \\
\end{bmatrix}
\]

\[ \text{REF} \]

pivot columns tell us which vectors among \( \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{w}_m \) to keep.

We will consider some examples next time.