Show your work for each problem.

1. Let $V$ be a vector space over a field $F$ and let $0$ be the zero vector in $V$. Below is a proof that $\alpha 0 = 0$ for any scalar $\alpha \in F$. Fill in the blanks with the axiom used at each step:

   proof. Let $\alpha$ be a scalar and let $-(\alpha 0)$ be the additive inverse of $\alpha 0$. Then
   
   \[ \alpha 0 = \alpha (0 + 0) \quad \text{A3: Zero vector} \]
   
   \[ = \alpha 0 + \alpha 0 \quad \text{D1: distribution over vector addition} \]

   Now add $-(\alpha 0)$ to both sides of the equation and simplify the left-hand side:
   
   \[ \alpha 0 + [-(\alpha 0)] = (\alpha 0 + \alpha 0) + [-(\alpha 0)] \]
   \[ \downarrow \quad \text{A4: Additive inverse} \]
   
   \[ 0 = (\alpha 0 + \alpha 0) + [-(\alpha 0)] \]

   We need three steps to simplify the right-hand side and complete the proof:

   \[ 0 = (\alpha 0 + \alpha 0) + [-(\alpha 0)] \]
   \[ \downarrow \quad \text{A2: associativity of addition} \]
   
   \[ 0 = \alpha 0 + (\alpha 0 + [-(\alpha 0)]) \]
   \[ \downarrow \quad \text{A4: Additive inverse} \]
   
   \[ 0 = \alpha 0 + 0 \]
   \[ \downarrow \quad \text{A3: Zero vector} \]
   
   \[ 0 = \alpha 0. \]

2. Let $V$ be a vector space. For any $v \in V$ let $-v$ denote the additive inverse of $v$. Prove that $-(-v) = v$ for any $v \in V$.

   (Hint: consider $v + [-v] + [-(-v)]$ and simplify in two different ways).

   Solution: Let $v \in V$. Then by associativity of addition we have
   
   \[ (v + [-v]) + [-(-v)] = v + ([-v] + [-(-v)]). \]

   The sums in each group of parentheses are sums of a vector and its additive inverse, so this simplifies to
   
   \[ 0 + [-(-v)] = v + 0. \]

   Finally, we use the fact that $0$ is the additive identity to conclude that
   
   \[ -(-v) = v. \]
3. Give an example of a non-empty subset $S$ of $\mathbb{R}^2$ that is closed under scalar multiplication (for all $x \in S$ and for all $c \in \mathbb{R}$, $cx \in S$), but that is not a subspace of $\mathbb{R}^2$.

**Solution:** There are many possible answers. Remember that a subset $S$ of a vector space is a subspace if it satisfies three axioms:

i) $S$ is closed under addition.

ii) $S$ is closed under scalar multiplication.

iii) $S$ contains 0.

One example is

$$S = \{(a, 0) : a \in \mathbb{R}\} \cup \{(0, b) : b \in \mathbb{R}\}$$

This set is not closed under addition, since $(1, 0), (0, 1) \in S$ but

$$(1, 0) + (0, 1) = (1, 1) \notin S.$$  

4. Let

$$V = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 = x_1 - x_2\}.$$  

a.) Prove that $V$ is a vector space over $\mathbb{R}$.

**Solution:** Since $V \subseteq \mathbb{R}$, we only need to show that $V$ is a subspace of $\mathbb{R}$. The three subspace axioms are

i) $S$ contains the zero vector.

ii) $S$ is closed under addition.

iii) $S$ is closed under scalar multiplication.

Notice that any element of $V$ can be written as $(x_1, x_2, x_1 - x_2)$ for $x_1, x_2 \in \mathbb{R}$.

Condition (i) is satisfied since $0 = 0 - 0$, so $(0, 0, 0) \in V$.

To prove (ii), let $(a, b, a - b)$ and $(c, d, c - d)$ be arbitrary elements of $V$. Then

$$(a, b, a - b) + (c, d, c - d) = (a + c, b + d, a - b + c - d)$$

$$= (a + c, b + d, (a + c) - (b + d)).$$

Since this sum satisfies the condition $x_3 = x_1 - x_2$, we have $(a, b, a - b) + (c, d, c - d) \in V$, so $V$ is closed under addition.

To prove (iii), let $(a, b, a - b)$ be an arbitrary element of $V$ and let $\lambda \in \mathbb{R}$. Then

$$\lambda(a, b, a - b) = (\lambda a, \lambda b, \lambda(a - b))$$

$$= (\lambda a, \lambda b, \lambda a - \lambda b).$$

Since this element satisfies the condition $x_3 = x_1 - x_2$, we have $\lambda(a, b, a - b) \in V$, so $V$ is closed under scalar multiplication.
b.) Find a basis for \( V \) (and prove that it is a basis). What is the dimension of \( V \)?

**Solution:** We can rewrite \( V \) as

\[
V = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 = x_1 - x_2\}
\]

\[
= \{(x_1, x_2, x_1 - x_2) : x_1, x_2, x_3 \in \mathbb{R}\}
\]

\[
= \{(x_1, 0, x_1) + (0, x_2, -x_2) : x_1, x_2, x_3 \in \mathbb{R}\}
\]

\[
= \{x_1(1, 0, 1) + x_2(0, 1, -1) : x_1, x_2, x_3 \in \mathbb{R}\}
\]

So every element of \( V \) can be written as a linear combination of \( v_1 = (1, 0, 1) \) and \( v_2 = (0, 1, -1) \). Notice that \( v_1 \) and \( v_2 \) each satisfy the condition \( x_3 = x_1 - x_2 \), so \( v_1, v_2 \in V \). These two facts together show that

\[
V = \text{Span}(\{v_1, v_2\}).
\]

To show \( B = \{v_1, v_2\} \) is a basis for \( V \), we must show that they are linearly independent. Let \( c_1, c_2 \in \mathbb{R} \) and consider the equation

\[
c_1v_1 + c_2v_2 = 0
\]

\[
\downarrow
\]

\[
c_1(1, 0, 1) + c_2(0, 1, -1) = (0, 0, 0)
\]

\[
\downarrow
\]

\[
(c_1, c_2, c_1 - c_2) = (0, 0, 0)
\]

In this last equation the first coordinate gives \( c_1 = 0 \) and the second coordinate gives \( c_2 = 0 \). Therefore \( B \) is linearly independent. Since \( B \) has exactly two elements, the dimension of \( V \) is 2.

5. The set \( \mathbb{R}^\mathbb{R} \) of functions \( f : \mathbb{R} \to \mathbb{R} \) with standard function addition and scalar multiplication forms a vector space over the field \( F = \mathbb{R} \). Determine whether the following sets of functions in \( \mathbb{R}^\mathbb{R} \) are linearly independent. If a set is linearly dependent, find a nontrivial linear combination equal to 0.

a.) \( \{e^x, e^{x^2}\} \)

**Solution:** Linearly dependent. Remember that

\[
e^{x^2 + 2} = e^x e^2 = e^2 \cdot e^x.
\]

Therefore

\[
-e^2(e^x) + (e^{x+2}) = -e^2 \cdot e^x + e^2 \cdot e^x = 0
\]

is a linear combination equal to 0 (\( c_1 = -e^2 \) and \( c_2 = 1 \)).
b.) \( \{ \cos^2(x), \sin^2(x) \} \)

**Solution:** **Linearly independent.** Let \( c_1, c_2 \in \mathbb{R} \) be constants and consider the equation
\[
c_1 \cos^2(x) + c_2 \sin^2(x) = 0
\]
where this is true for all \( x \in \mathbb{R} \). To prove that the functions are linearly independent, we must show that \( c_1 = c_2 = 0 \).

If \( x = 0 \) then the equation becomes
\[
c_1(1) + c_2(0) = 0 \quad \rightarrow \quad c_1 = 0.
\]

If \( x = \pi/2 \) then the equation becomes
\[
c_1(0) + c_2(1) = 0 \quad \rightarrow \quad c_2 = 0.
\]

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c.) \( \{ \cos^2(x), \sin^2(x), 5 \} \)

**Solution:** **Linearly dependent.** Remember the trigonometric identity
\[
\cos^2(x) + \sin^2(x) = 1.
\]

Subtract 1 from both sides to get
\[
\cos^2(x) + \sin^2(x) - 1 = 0.
\]

Now we just need to write \(-1\) as \( (-\frac{1}{5})5 \):
\[
\boxed{\cos^2(x) + \sin^2(x) + (-\frac{1}{5})5} = 0
\]

\( (c_1 = 1, c_2 = 1, \text{ and } c_3 = -\frac{1}{5}) \).